On a Conjecture of Rapoport and Zink

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Abstract

In their book Rapoport and Zink constructed rigid analytic period spaces $\mathcal{F}^{wa}$ for Fontaine's filtered isocrystals and period morphisms from moduli spaces of p-divisible groups to some of these period spaces. They conjectured the existence of an étale bijective morphism $\mathcal{F}^a \to \mathcal{F}^{wa}$ of rigid analytic spaces and of interesting local systems of $\mathbb{Q}_p$-vector spaces on $\mathcal{F}^a$. For those period spaces possessing period morphisms de Jong pointed out that one may take $\mathcal{F}^a$ as the image of the period morphism, viewed as a morphism of Berkovich spaces, and take the rational Tate module of the universal p-divisible group as the desired local system on $\mathcal{F}^a$. In this article we construct for Hodge-Tate weights 0 and 1 an intrinsic Berkovich open subspace of $\mathcal{F}^{wa}$ through which the period morphism factors and which we conjecture to be the image of the period morphism. We present indications supporting our conjecture and we show that only in exceptional cases our open subspace equals all of $\mathcal{F}^{wa}$.
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Introduction

Rapoport and Zink fix an F-isocrystal (D, φ_D) over $\mathbb{F}_p^{\mathrm{alg}}$ and consider the partial flag variety $\breve{\mathcal{F}}$ over $K_0 := W(\mathbb{F}_p^{\mathrm{alg}})[\frac{1}{p}]$ parametrizing filtrations of D with fixed Hodge-Tate weights. They show that the set of weakly admissible filtrations is a rigid analytic subspace $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ of $\breve{\mathcal{F}}$, which is called the *period space*; see [28, Proposition 1.36]. They conjecture the existence of an étale morphism $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ of rigid analytic spaces which is bijective on rigid analytic points, and of a local system of $\mathbb{Q}_p$-vector spaces on $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}}$ such that at every point μ of $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}}$ the fiber of the local system is the crystalline Galois representation corresponding by the Colmez-Fontaine Theorem [12] to the filtration on (D, φ_D) given by the image of μ in $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$. We recall the precise formulation of the conjecture and the definition of $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ in Section 1. De Jong [22] pointed out that to study local systems it is best to work in the category of Berkovich's K_0-analytic spaces rather than rigid analytic spaces. For convenience of the reader we review Berkovich's definition and the relation with rigid analytic spaces in Appendix A. We show that in fact $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ is the rigid analytic space associated with an open K_0-analytic subspace $\breve{\mathcal{F}}_b^{wa}$ of $\breve{\mathcal{F}}$ (Proposition 1.3).

If the Hodge-Tate weights all are 0 or 1 we present in this article an open K_0-analytic subspace $\breve{\mathcal{F}}_b^a$ of $\breve{\mathcal{F}}_b^{wa}$ whose associated rigid analytic space $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}}$ is a candidate for the

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space searched for in Rapoport and Zink's conjecture. The construction of $\breve{\mathcal{F}}_b^a$ is as follows. With any analytic point μ of $\breve{\mathcal{F}}$ (these are the ones of which K_0-analytic spaces consist; see Definition A.1) we associate in Section 4 a φ-module $\mathbf{M}_\mu$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and we let $\breve{\mathcal{F}}_b^a$ be the set of those μ for which $\mathbf{M}_\mu$ is isoclinic of slope zero. The ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is defined in Section 2 and the notion of φ-modules is recalled in Section 3. Our construction is inspired by Berger's [4] construction which associates with any *rigid analytic point* $\mu \in \breve{\mathcal{F}}$ (the ones whose residue field $\mathcal{H}(\mu)$ is *finite* over K_0) a (φ, Γ)-module over the Robba ring. Due to the restriction to rigid analytic points Berger's approach works even without the above assumption on the Hodge-Tate weights. However, the techniques we use in Section 5 where we prove that $\breve{\mathcal{F}}_b^a$ is an open K_0-analytic subspace of $\breve{\mathcal{F}}$ require in an essential way to work with K_0-analytic spaces, which are topological spaces in the classical sense, and not rigid analytic spaces. Namely we show that the complement $\breve{\mathcal{F}} \smallsetminus \breve{\mathcal{F}}_b^a$ is the image of a compact set under a continuous map (Theorem 5.2). It is worth noting that this crude map is in fact only continuous and not a morphism of K_0-analytic or rigid analytic spaces. We also show that only in rare cases the inclusion $\breve{\mathcal{F}}_b^a \subset \breve{\mathcal{F}}_b^{wa}$ is an equality (Remark 5.5). The fact that this inclusion may be strict was noted in [28] and [22] as a peculiarity. We give a natural explanation for this phenomenon.

For Hodge-Tate weights 0 and 1 Rapoport and Zink also study a period morphism from the moduli space $\mathcal{M}^{\mathrm{an}}$ of p-divisible groups with additional structure to $\breve{\mathcal{F}}_b^{wa}$. The problem to determine the image of this period morphism was already mentioned by Grothendieck [19]. We prove in Section 6 that the period morphism factors through our set $\breve{\mathcal{F}}_b^a$. We conjecture that $\breve{\mathcal{F}}_b^a$ is its image and, moreover, that $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}}$ and the rational Tate module of the universal p-divisible group over $\mathcal{M}^{\mathrm{an}}$ are the space and the local system searched for in the conjecture of Rapoport and Zink. We present our motivation for this conjecture in Section 7. The fact that the inclusion $\breve{\mathcal{F}}_b^a \subset \breve{\mathcal{F}}_b^{wa}$ may be strict while it induces a bijection on rigid analytic points by the Colmez-Fontaine Theorem [12] again demonstrates the necessity to work with Berkovich's K_0-analytic spaces instead of rigid analytic spaces.

To end this introduction let us mention that the ideas in this article are inspired by our analogous theory in equal characteristic [21] where we were able to prove the analogue of the Rapoport-Zink Conjecture and the surjectivity of the period morphism onto $\breve{\mathcal{F}}_b^a$. In the beginning we had hoped to be able to extend our approach here also to Hodge-Tate weights other than 0 and 1, but we did not succeed. Not even the construction of the φ-module $\mathbf{M}_\mu$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$, let alone the openness result could be established along this line. The reason lies in the fact that for analytic points $\mu \in \breve{\mathcal{F}}$ whose residue field $\mathcal{H}(\mu)$ is not finite over K_0 the ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is not a $\mathcal{H}(\mu)$-algebra (like Fontaine's field $\mathbf{B}_{\mathrm{dR}}$ is not a $\mathbb{C}_p$-algebra). So Berger's construction could not be adapted.

1 The Conjecture of Rapoport and Zink

In this section we explain the construction of Rapoport and Zink's p-adic period spaces and their conjecture mentioned above. Let us first recall the definition of filtered isocrystals. We denote by $\mathbb{F}_p^{\mathrm{alg}}$ an algebraic closure of the finite field with p elements and by $K_0 := W(\mathbb{F}_p^{\mathrm{alg}})[\frac{1}{p}]$ the fraction field of the ring of Witt vectors over $\mathbb{F}_p^{\mathrm{alg}}$. Let $\varphi = W(\mathrm{Frob}_p)$ be the Frobenius lift on K_0.

Definition 1.1. An *F-isocrystal over* $\mathbb{F}_p^{\mathrm{alg}}$ is a finite dimensional K_0-vector space D equipped with a φ-linear endomorphism φ_D. If K is a field extension of K_0 and $Fil^\bullet D_K$

is an exhaustive separated decreasing filtration of $D_K := D \otimes_{K_0} K$ by K-subspaces we say that $\underline{D} = (D, \varphi_D, Fil^\bullet D_K)$ is a *filtered isocrystal over* K. The integers h for which $Fil^{-h} D_K \neq Fil^{-h+1} D_K$ are called the *Hodge-Tate weights* of $\underline{D}$. We let $t_N(\underline{D})$ be the p-adic valuation of $\det \varphi_D$ (with respect to any basis of D) and we let

$$t_H(\underline{D}) \;=\; \sum_{i \in \mathbb{Z}} i \cdot \dim_K gr^i_{Fil^\bullet}(D_K)\,.$$

The filtered isocrystal $\underline{D}$ is called *weakly admissible*[1] if

$$t_H(\underline{D}) \;=\; t_N(\underline{D}) \qquad \text{and} \qquad t_H(\underline{D}') \;\leq\; t_N(\underline{D}') \tag{1.1}$$

for any subobject $\underline{D}' = \big(D', \varphi_D|_{D'}, Fil^\bullet D'_K\big)$ of $\underline{D}$, where D' is any φ_D-stable K_0-subspace of D equipped with the induced filtration $Fil^\bullet D'_K$ on $D'_K := D' \otimes_{K_0} K$.

To construct period spaces let G be a reductive linear algebraic group over $\mathbb{Q}_p$. Fix a conjugacy class $\{\mu\}$ of cocharacters

$$\mu : \mathbb{G}_m \to G$$

defined over subfields of $\mathbb{C}_p$ (the p-adic completion of an algebraic closure of K_0). Let E be the field of definition of the conjugacy class. It is a finite extension of $\mathbb{Q}_p$. Two cocharacters in this conjugacy class are called *equivalent* if they induce the same weight filtration on the category $\mathrm{Rep}_{\mathbb{Q}_p} G$ of finite dimensional $\mathbb{Q}_p$-rational representations of G. There is a projective variety $\mathcal{F}$ over E whose $\mathbb{C}_p$-valued points are in bijection with the equivalence classes of cocharacters (from the fixed conjugacy class). Namely let V in $\mathrm{Rep}_{\mathbb{Q}_p} G$ be any faithful representation of G. With a cocharacter μ defined over a field K one associates the filtration $Fil^i_\mu V_K := \bigoplus_{j \geq i} V_{K,j}$ of $V_K := V \otimes_{\mathbb{Q}_p} K$ given by the weight spaces

$$V_{K,j} \;:=\; \big\{\, v \in V_K : \;\; \mu(z) \cdot v \;=\; z^j v \text{ for all } z \in \mathbb{G}_m(K) \,\big\}\,.$$

This defines a closed embedding of $\mathcal{F}$ into a partial flag variety of V

$$\mathcal{F} \;\hookrightarrow\; \mathcal{F}lag(V) \otimes_{\mathbb{Q}_p} E\,, \tag{1.2}$$

where the points of $\mathcal{F}lag(V)$ with value in a $\mathbb{Q}_p$-algebra R are the filtrations F^i of $V \otimes_{\mathbb{Q}_p} R$ by R-submodules which are direct summands such that $\mathrm{rk}_R\, gr^i_{F^\bullet}$ is the multiplicity of the weight i of the conjugacy class $\{\mu\}$ on V.

Now let $b \in G(K_0)$. For any V in $\mathrm{Rep}_{\mathbb{Q}_p} G$ one obtains an F-isocrystal

$$(V \otimes_{\mathbb{Q}_p} K_0\,,\; b\cdot\varphi)\,.$$

A pair (μ, b) with a cocharacter $\mu : \mathbb{G}_m \to G$ defined over $K \supset K_0$ and an element $b \in G(K_0)$ is called *weakly admissible* if for some faithful representation V in $\mathrm{Rep}_{\mathbb{Q}_p} G$ the filtered isocrystal

$$\mathcal{I}(V) \;:=\; (V \otimes_{\mathbb{Q}_p} K_0\,,\; b\cdot\varphi\,,\; Fil^\bullet_\mu V_K)$$

is weakly admissible. In fact this then holds for any V in $\mathrm{Rep}_{\mathbb{Q}_p} G$; see [28, 1.18]. If (μ, b) is weakly admissible and K/K_0 is finite then the filtered isocrystal $\mathcal{I}(V)$ is *admissible*, that is

[1] This used to be the terminology until Colmez–Fontaine [12] showed that for K/K_0 finite *weakly admissible implies admissible*. Since we consider also infinite extensions K/K_0 for which the Colmez–Fontaine Theorem fails we stick to the old terminology.

it arises from a crystalline Galois-representation $\rho : \mathrm{Gal}(K^{\mathrm{alg}}/K) \to \mathrm{GL}(U)$ via Fontaine's functor

$$\mathbf{D}_{\mathrm{cris}}(U) \;:=\; (U \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{\mathrm{Gal}(K^{\mathrm{alg}}/K)}$$

which by the Colmez-Fontaine theorem [12] is an equivalence of categories from crystalline representations of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ to weakly admissible filtered isocrystals over K. The assignment

$$\mathrm{Rep}_{\mathbb{Q}_p} G \;\longrightarrow\; (\mathbb{Q}_p\text{vector spaces})\,, \quad V \;\mapsto\; U \tag{1.3}$$

defines a fiber functor on $\mathrm{Rep}_{\mathbb{Q}_p} G$.

Let $\breve{E} = EK_0$ be the completion of the maximal unramified extension of E. In what follows we consider cocharacters μ defined over finite extensions K of $\breve{E}$. Let $\breve{\mathcal{F}}^{\mathrm{rig}}$ be the rigid analytic space over $\breve{E}$ associated with the variety $\breve{\mathcal{F}} = \mathcal{F} \otimes_E \breve{E}$. Rapoport and Zink define the *p-adic period space* associated with $(G, b, \{\mu\})$ as

$$(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}} \;:=\; \big\{\, \mu \in \breve{\mathcal{F}}^{\mathrm{rig}} : \;\; (\mu, b) \text{ is weakly admissible} \,\big\}\,.$$

They show that $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ is an admissible open rigid analytic subspace of $\breve{\mathcal{F}}^{\mathrm{rig}}$; see [28, Proposition 1.36]. Namely they reduce to the case where b is *decent with integer* s, that is for some positive integer s and some cocharacter $sv : \mathbb{G}_m \to G$ defined over K_0

$$b \cdot \varphi(b) \cdot \ldots \cdot \varphi^{s-1}(b) \;=\; sv(p)\,.$$

(In fact sv is the s-fold multiple of the slope cocharacter v; see [28, 1.7].) If b is decent with integer s then $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ has a natural structure of rigid analytic subspace of $(\mathcal{F} \otimes_E E_s)^{\mathrm{rig}}$ over $E_s = E\,W(\mathbb{F}_{p^s})[\frac{1}{p}]$ from which it arises by base change to $\breve{E}$. Rapoport and Zink make the

Conjecture 1.2. *([28, p. 29]) There exists an étale morphism $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ of rigid analytic spaces over $\breve{E}$ which is bijective on (rigid analytic) points and there exists a tensor functor from $\mathrm{Rep}_{\mathbb{Q}_p} G$ to the category $\mathbb{Q}_p$-$\underline{\mathrm{Loc}}_{(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}}$ of local systems of $\mathbb{Q}_p$-vector spaces on $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}$ with the following property:*

> *For any point $\mu \in (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}(K)$ with $K/\breve{E}$ finite, the fiber functor which associates with a representation in $\mathrm{Rep}_{\mathbb{Q}_p} G$ the fiber at μ of the corresponding local system is isomorphic to the fiber functor (1.3).*

(1.4)

The notion of local systems of $\mathbb{Q}_p$-vector spaces on rigid analytic spaces was studied by de Jong [22, §4] who pointed out that this is best done working with Berkovich's $\breve{E}$-analytic spaces rather than rigid analytic spaces. So let $\breve{\mathcal{F}}^{\mathrm{an}}$ be the $\breve{E}$-analytic space associated with the variety $\breve{\mathcal{F}}$. See Appendix A for the notion of $\breve{E}$-analytic space and the relation with rigid analytic spaces.

Proposition 1.3. *There exists an open $\breve{E}$-analytic subspace $\breve{\mathcal{F}}_b^{wa}$ of $\breve{\mathcal{F}}^{\mathrm{an}}$ whose associated rigid analytic space is the period space $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$.*

Proof. From its construction in [28, Proposition 1.34] the period space $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ possesses an admissible covering $\{X_i^{\mathrm{rig}}\}_{i\in\mathbb{N}}$ by admissible subsets $X_i^{\mathrm{rig}} \subset \breve{\mathcal{F}}^{\mathrm{rig}}$ such that $\breve{\mathcal{F}}^{\mathrm{rig}} \smallsetminus X_i^{\mathrm{rig}}$ is a finite union $\bigcup_j \mathrm{Sp}\, B_{i,j}$ of affinoid subdomains and $X_i^{\mathrm{rig}} \subset X_{i+1}^{\mathrm{rig}}$. The X_i^{rig} correspond to

open $\breve{E}$-analytic subspaces $X_i := \breve{\mathcal{F}}^{\mathrm{an}} \smallsetminus \bigcup_j \mathcal{M}(B_{i,j})$ of $\breve{\mathcal{F}}^{\mathrm{an}}$. Moreover, it follows from [28, Proposition 1.34] that the closure $\overline{X}_i$ of X_i in $\breve{\mathcal{F}}^{\mathrm{an}}$ is a finite union of $\breve{E}$-affinoid subdomains and $\bar{X}_i \subset X_{i+1}$. Thus the union $\breve{\mathcal{F}}_b^{wa} := \bigcup_{i\in\mathbb{N}} X_i$ is an open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{\mathrm{an}}$ whose associated rigid analytic space equals $(\breve{\mathcal{F}}^{wa})^{\mathrm{rig}}$. (Use [6, Lemma 1.6.2] to see that the Grothendieck topology induced from $\breve{\mathcal{F}}_b^{wa}$ on $\{\, \mu \in \breve{\mathcal{F}}_b^{wa} :\ \mathcal{H}(\mu)/\breve{E} \text{ is finite} \,\}$ coincides with the Grothendieck topology of $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$.) $\square$

After recalling some of Fontaine's rings and some facts on φ-modules in the next two sections we will construct in Sections 4 and 5 for Hodge-Tate weights 0 and 1 an open $\breve{E}$-analytic subspace $\breve{\mathcal{F}}_b^{a}$ of $\breve{\mathcal{F}}_b^{wa}$ such that $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}$ is a candidate for the space searched for in Conjecture 1.2.

Let us also mention the following fact for local systems on an $\breve{E}$-analytic space X.

Proposition 1.4. *([22, Theorem 4.2].) For any geometric point $\bar{x}$ of X there is a natural $\mathbb{Q}_p$-linear tensor functor*

$$w_{\bar{x}} :\ \mathbb{Q}_p\text{-}\underline{\mathrm{Loc}}_X \ \to\ \underline{\mathrm{Rep}}_{\mathbb{Q}_p}\big(\pi_1^{\text{ét}}(X,\bar{x})\big)$$

which assigns to a local system $\mathcal{V}$ the $\pi_1^{\text{ét}}(X,\bar{x})$-representation $\mathcal{V}_{\bar{x}}$. It is an equivalence if X is connected.

Here $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}\big(\pi_1^{\text{ét}}(X,\bar{x})\big)$ is the category of continuous $\mathbb{Q}_p$-linear representations of the étale fundamental group $\pi_1^{\text{ét}}(X,\bar{x})$ of X which was defined by de Jong [22].

2 Fontaine's Rings

We recall from Colmez [11] some of the rings used in p-adic Hodge theory. Let $\mathcal{O}_K$ be a complete valuation ring of rank one which is an extension of $\mathbb{Z}_p$ and let K be its fraction field. Let v_p be the valuation on $\mathcal{O}_K$ which we assume to be normalized so that $v_p(p) = 1$. In p-adic Hodge theory it is usually assumed that $\mathcal{O}_K$ is discretely valued with perfect residue field. However, in Section 4 we need the more general situation introduced here. Let C be the completion of an algebraic closure K^{alg} of K and let $\mathcal{O}_C$ be the valuation ring of C. For $n \in \mathbb{N}$ let $\varepsilon^{(n)}$ be a primitive p^n-th root of unity chosen in such a way that $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ for all n. We define

$$\widetilde{\mathbf{E}}^+ \ := \ \widetilde{\mathbf{E}}^+(C) \ := \ \big\{\, x = (x^{(n)})_{n\in\mathbb{N}_0} :\ x^{(n)} \in \mathcal{O}_C\,,\ (x^{(n+1)})^p = x^{(n)} \,\big\}\,.$$

Fix the elements $\varepsilon := (1, \varepsilon^{(1)}, \varepsilon^{(2)}, \ldots)$ and $\overline{\pi} := \varepsilon - 1$ of $\widetilde{\mathbf{E}}+$. With the multiplication $xy := \big(x^{(n)}y^{(n)}\big)_{n\in\mathbb{N}_0}$, the addition $x + y := \big(\lim_{m\to\infty}(x^{(m+n)} + y^{(m+n)})^{p^m}\big)_{n\in\mathbb{N}_0}$, and the valuation $v_{\mathbf{E}}(x) := v_p(x^{(0)})$, $\widetilde{\mathbf{E}}^+(C)$ becomes a complete valuation ring of rank one with algebraically closed fraction field, called $\widetilde{\mathbf{E}} := \widetilde{\mathbf{E}}(C)$, of characteristic p. Next we define

$$\begin{aligned}
\widetilde{\mathbf{A}}^+ &:= \widetilde{\mathbf{A}}^+(C) := W\big(\widetilde{\mathbf{E}}^+(C)\big) \text{ and}\\
\widetilde{\mathbf{A}} &:= \widetilde{\mathbf{A}}(C) := W\big(\widetilde{\mathbf{E}}(C)\big) \text{ the rings of Witt vectors,}\\
\widetilde{\mathbf{B}}^+ &:= \widetilde{\mathbf{B}}^+(C) := \widetilde{\mathbf{A}}^+(C)[\tfrac{1}{p}] \text{ and}\\
\widetilde{\mathbf{B}} &:= \widetilde{\mathbf{B}}(C) := \widetilde{\mathbf{A}}(C)[\tfrac{1}{p}] \text{ the fraction field of } \widetilde{\mathbf{A}}(C).
\end{aligned}$$

By Witt vector functoriality there is the Frobenius lift $\varphi := W(\mathrm{Frob}_p)$ on the later rings. For $x \in \widetilde{\mathbf{E}}(C)$ we let $[x] \in \widetilde{\mathbf{A}}(C)$ denote the Teichmüller lift. Set $\pi := [\varepsilon] - 1$. If $x = \sum_{i \gg -\infty}^{\infty} p^i [x_i] \in \widetilde{\mathbf{A}}(C)$ then we set $w_k(x) := \min\{ v_{\mathbf{E}}(x_i) : i \leq k \}$. For $r > 0$ let

$$\begin{aligned}
\widetilde{\mathbf{A}}^{(0,r]} &:= \widetilde{\mathbf{A}}^{(0,r]}(C) := \left\{ x \in \widetilde{\mathbf{A}}(C) : \lim_{k \to +\infty} w_k(x) + \tfrac{k}{r} = +\infty \right\}, \\
\widetilde{\mathbf{B}}^{(0,r]} &:= \widetilde{\mathbf{B}}^{(0,r]}(C) := \widetilde{\mathbf{A}}^{(0,r]}(C)[\tfrac{1}{p}], \\
\widetilde{\mathbf{B}}^{\dagger} &:= \widetilde{\mathbf{B}}^{\dagger}(C) := \bigcup_{0<r} \widetilde{\mathbf{B}}^{(0,r]}(C).
\end{aligned}$$

One has $\widetilde{\mathbf{A}}^{+}(C) \subset \widetilde{\mathbf{A}}^{(0,r]}(C)$ for all $r > 0$. On $\widetilde{\mathbf{B}}^{(0,r]}(C)$ there is a valuation defined for $x = \sum_{i \gg -\infty}^{\infty} p^i [x_i]$ as

$$v^{(0,r]}(x) := \min\{ w_k(x) + \tfrac{k}{r} : k \in \mathbb{Z} \} = \min\{ v_{\mathbf{E}}(x_i) + \tfrac{i}{r} : i \in \mathbb{Z} \}.$$

Now one defines $\widetilde{\mathbf{B}}^{]0,r]}(C)$ as the Fréchet completion of $\widetilde{\mathbf{B}}^{(0,r]}(C)$ with respect to the family of semi-valuations $v^{[s,r]}(x) := \min\{ v^{(0,s]}(x), v^{(0,r]}(x) \}$ for $0 < s \leq r$. This means in concrete terms that a sequence of elements $x_n \in \widetilde{\mathbf{B}}^{(0,r]}(C)$ converges in $\widetilde{\mathbf{B}}^{]0,r]}(C)$ if and only if $\lim_{n\to\infty} v^{[s,r]}(x_{n+1} - x_n) = +\infty$ for all $0 < s \leq r$. Also if $r \geq s$ we let $\widetilde{\mathbf{B}}^{[s,r]}(C)$ be the completion of $\widetilde{\mathbf{B}}^{(0,r]}(C)$ with respect to $v^{[s,r]}$. We view $\widetilde{\mathbf{B}}^{]0,r]}(C)$ as a subring of $\widetilde{\mathbf{B}}^{I}(C)$ for any closed subinterval $I \subset (0,r]$. If $I =]0,r]$ or $I = [s,r]$ the functions

$$f_i : \widetilde{\mathbf{B}}^{(0,r]}(C) \longrightarrow \widetilde{\mathbf{E}}(C) \quad \text{defined by} \quad x = \sum_{i \gg -\infty}^{\infty} p^i [f_i(x)] \tag{2.1}$$

extend by continuity to $\widetilde{\mathbf{B}}^{I}(C)$ and for any $x \in \widetilde{\mathbf{B}}^{I}(C)$ the sum $\sum_{i=-\infty}^{\infty} p^i [f_i(x)]$ converges to x in $\widetilde{\mathbf{B}}^{I}(C)$; see [24, 2.5.1]. Let

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} := \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C) := \bigcup_{r>0} \widetilde{\mathbf{B}}^{]0,r]}(C).$$

The morphism φ extends to bicontinuous isomorphisms of rings $\widetilde{\mathbf{B}}^{]0,pr]}(C) \to \widetilde{\mathbf{B}}^{]0,r]}(C)$ and $\widetilde{\mathbf{B}}^{[ps,pr]}(C) \to \widetilde{\mathbf{B}}^{[s,r]}(C)$ defining an automorphism of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$. The rings $\widetilde{\mathbf{B}}^{]0,r]}(C)$ and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ were studied in detail by Berger [3, §2] who called the first $\widetilde{\mathbf{B}}^{\dagger,(p-1)/pr}_{\mathrm{rig}}$ instead. Note that the ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ is denoted $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ by Kedlaya [24].

There is a homomorphism $\theta : \widetilde{\mathbf{B}}^{(0,1]}(C) \to C$ sending $\sum_{i \gg -\infty}^{\infty} p^i [x_i]$ to $\sum_{i \gg -\infty}^{\infty} p^i x_i^{(0)}$ which extends by continuity to $\widetilde{\mathbf{B}}^{]0,1]}(C)$. The series

$$t := \log[\varepsilon] = \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \pi^n \in \widetilde{\mathbf{B}}^{]0,1]}(C)$$

generates $\ker\theta$. Let

$\mathbf{B}^{+}_{\mathrm{cris}} := \mathbf{B}^{+}_{\mathrm{cris}}(C)$ be the p-adic completion of $\widetilde{\mathbf{B}}^{+}(C)\big[\frac{w^n}{n!} : w \in \ker\theta, n \in \mathbb{N}\big]$ and

$\mathbf{B}_{\mathrm{cris}} := \mathbf{B}_{\mathrm{cris}}(C) := \mathbf{B}^{+}_{\mathrm{cris}}(C)[\frac{1}{t}]$.

The morphism φ extends to endomorphisms of $\mathbf{B}^{+}_{\mathrm{cris}}(C)$ and $\mathbf{B}_{\mathrm{cris}}(C)$. Let

$$\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} := \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(C) := \bigcap_{n \in \mathbb{N}_0} \varphi^n \mathbf{B}^{+}_{\mathrm{cris}}(C).$$

One easily sees that $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(C) \subset \widetilde{\mathbf{B}}^{]0,r]}(C)$ for any $r > 0$. More precisely, $\widetilde{\mathbf{B}}^{]0,r]}(C)$ equals the p-adic completion of $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(C)[\frac{p}{[\overline{\pi}]}]$ and is hence a flat $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(C)$-algebra.

3 φ-Modules

Let K and C be as in the previous section. To shorten notation we will drop the denotation (C) from the rings introduced there. We recall some definitions and facts from Kedlaya [24]. Let a be a positive integer.

Definition 3.1. A *φ^a-module* over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is a finite free $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module $\mathbf{M}$ with a φ^a-semilinear map $\varphi_{\mathbf{M}}: \mathbf{M} \to \mathbf{M}$ such that the image of $\varphi_{\mathbf{M}}$ generates $\mathbf{M}$ as $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module. The rank of $\mathbf{M}$ as a $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module is denoted $\mathrm{rk}\,\mathbf{M}$. A *morphism of φ^a-modules* is a morphism of the underlying $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-modules which commutes with the $\varphi_{\mathbf{M}}$'s. We denote the set of morphism between two φ^a-modules $\mathbf{M}$ and $\mathbf{M}'$ by $\mathrm{Hom}_{\varphi^a}(\mathbf{M},\mathbf{M}')$.

The category of φ^a-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ possesses tensor products and duals. For a positive integer b there is a *restriction of Frobenius functor* $[b]_*$ from φ^a-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ to φ^{ab}-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ sending $(\mathbf{M},\varphi_{\mathbf{M}})$ to $(\mathbf{M},\varphi^b_{\mathbf{M}})$.

Example 3.2. Let $c,d\in\mathbb{Z}$ with $d>0$ and $(c,d)=1$. Define the φ^a-module $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ as $\mathbf{M}(c,d)=\bigoplus_{i=1}^d \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\mathbf{e}_i$ equipped with

$$\varphi_{\mathbf{M}}(\mathbf{e}_1) \;=\; \mathbf{e}_2\,, \quad \ldots\,, \quad \varphi_{\mathbf{M}}(\mathbf{e}_{d-1}) \;=\; \mathbf{e}_d\,, \quad \varphi_{\mathbf{M}}(\mathbf{e}_d) \;=\; p^c\mathbf{e}_1\,. \tag{3.1}$$

Lemma 3.3. *([24, Lemmas 4.1.2 and 3.2.4]) The φ^a-modules $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfy*

(a) $\mathbf{M}(c,d)^\vee \;\cong\; \mathbf{M}(-c,d)$ *and*

(b) $[d]_*\mathbf{M}(c,d) \;\cong\; \mathbf{M}(c,1)^{\oplus d}$.

Definition 3.4. For a φ^a-module $\mathbf{M}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ we define the set of *φ^a-invariants* as

$$\mathrm{H}^0_{\varphi^a}(\mathbf{M}) \;:=\; \{\, x\in\mathbf{M}:\; \varphi_{\mathbf{M}}(x)=x \,\}\,.$$

It is a vector space over $\mathrm{H}^0_{\varphi^a}\big(\mathbf{M}(0,1)\big) \;=\; W(\mathbb{F}_{p^a})[p^{-1}]$; use [24, Proposition 3.3.4]. We have $\mathrm{Hom}_{\varphi^a}(\mathbf{M},\mathbf{M}')=\mathrm{H}^0_{\varphi^a}(\mathbf{M}^\vee\otimes\mathbf{M}')$ for φ^a-modules $\mathbf{M}$ and $\mathbf{M}'$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.

Proposition 3.5. *([24, 4.1.3 and 4.1.4]) The φ^a-modules $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfy*

(a) $\mathrm{Hom}_{\varphi^a}\big(\mathbf{M}(c,d),\mathbf{M}(c',d')\big) \;\neq\; (0)$ *if and only if* $c/d\geq c'/d'$.

(b) $\mathrm{H}^0_{\varphi^a}\big(\mathbf{M}(c,d)\big) \;\neq\; (0)$ *if and only if* $c/d\leq 0$.

Proposition 3.6. *If $c>0$ then the φ^a-module $\mathbf{M}(-c,1)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfies*

$$\mathrm{H}^0_{\varphi^a}\big(\mathbf{M}(-c,1)\big) \;=\; \Big\{\sum_{\nu\in\mathbb{Z}} p^{c\nu}\sum_{j=0}^{c-1} p^j\varphi^{-a\nu}([x_j]):\; x_0,\ldots,x_{c-1}\in\widetilde{\mathbf{E}}\,, v_{\mathbf{E}}(x_j)>0\Big\}\,.$$

Proof. One easily verifies that the series on the right converges (even in $\widetilde{\mathbf{B}}^{]0,r]}$ for all $r>0$) and is a φ^a-invariant of $\mathbf{M}(-c,1)$.

Conversely let $x\in\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfy $x=p^{-c}\varphi^a(x)$. There is an $r>0$ with $x\in\widetilde{\mathbf{B}}^{]0,r]}$. Let $f_i:\widetilde{\mathbf{B}}^{]0,r]}\to\widetilde{\mathbf{E}}$ be the functions from (2.1) which satisfy $x=\sum_{i=-\infty}^{\infty}p^i[f_i(x)]$. The relation

$x = p^{-c}\varphi^a(x)$ yields $f_i(x)^{p^a} = f_{i-c}(x)$. So x is determined by $x_j := f_j(x)$ for $j = 0, \ldots, c-1$. Since $x \in \widetilde{\mathbf{B}}^{]0,r]}$ we must have $\lim_{i\to-\infty} v^{(0,r]}\big(p^i[f_i(x)]\big) = \infty$. For $i = j - kc$ we compute

$$v^{(0,r]}\big(p^i[f_i(x)]\big) \;=\; v_{\mathbf{E}}\big(f_i(x)\big) + \tfrac{i}{r} \;=\; p^{ak}v_{\mathbf{E}}(x_j) + \tfrac{j-kc}{r}$$

and so we must have $v_{\mathbf{E}}(x_j) > 0$ for all j. □

For φ^a-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ Kedlaya proved the following structure theorem.

Theorem 3.7. *([24, Theorem 4.5.7]) Any φ^a-module $\mathbf{M}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is isomorphic to a direct sum of φ^a-modules $\mathbf{M}(c_i, d_i)$ for uniquely determined pairs (c_i, d_i) up to permutation. It satisfies $\wedge^d\mathbf{M} \cong \mathbf{M}(c, 1)$ where $c = \sum_i c_i$ and $d = \operatorname{rk}\mathbf{M} = \sum_i d_i$.*

Definition 3.8. If $\det\mathbf{M} := \wedge^{\operatorname{rk}\mathbf{M}}\mathbf{M} \cong \mathbf{M}(c, 1)$ we define the *degree* and the *weight* of $\mathbf{M}$ as $\deg\mathbf{M} := c$ and $\operatorname{wt}\mathbf{M} := \frac{\deg\mathbf{M}}{\operatorname{rk}\mathbf{M}}$.

Proposition 3.9. *([24, Lemma 3.4.9]) Every φ^a-submodule $\mathbf{M}' \subset \mathbf{M}(c, d)^{\oplus n}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfies $\operatorname{wt}\mathbf{M}' \geq \operatorname{wt}\mathbf{M}(c, d)^{\oplus n} = \frac{c}{d}$.*

Let us end this section with a remark on radii of convergence. Let $\mathbf{M}^{]0,r]}$ be a free $\widetilde{\mathbf{B}}^{]0,r]}$-module and for $0 < s \leq r$ let $\mathbf{M}^{]0,s]} := \mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\iota} \widetilde{\mathbf{B}}^{]0,s]}$ be obtained by base change via the natural inclusion $\iota : \widetilde{\mathbf{B}}^{]0,r]} \hookrightarrow \widetilde{\mathbf{B}}^{]0,s]}$. Let further

$$\varphi_{\mathbf{M}}^{]0,rp^{-a}]} :\; \mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\varphi^a} \widetilde{\mathbf{B}}^{]0,rp^{-a}]} \;\xrightarrow{\;\sim\;}\; \mathbf{M}^{]0,rp^{-a}]}$$

be an isomorphism. We say that a φ^a-module $(\mathbf{M}, \varphi_{\mathbf{M}})$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is *represented* by the pair $(\mathbf{M}^{]0,r]}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]})$ if $(\mathbf{M}, \varphi_{\mathbf{M}}) = (\mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]}} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]} \otimes \mathrm{id})$.

Proposition 3.10. *If $\mathbf{M}$ is represented by $(\mathbf{M}^{]0,r]}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]})$ then*

$$\mathrm{H}^0_{\varphi^a}(\mathbf{M}) \;=\; \big\{\, x \in \mathbf{M}^{]0,r]} :\; \varphi_{\mathbf{M}}^{]0,rp^{-a}]}(x \otimes_{\varphi^a} 1) = x \otimes_\iota 1 \,\big\}\,.$$

Proof. The inclusion "$\supset$" follows from the inclusion $\mathbf{M}^{]0,r]} \subset \mathbf{M}$.

To prove the opposite inclusion "$\subset$" let $x \in \mathrm{H}^0_{\varphi^a}(\mathbf{M})$. Since $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} = \bigcup_{s>0} \widetilde{\mathbf{B}}^{]0,s]}$ there exists a $0 < s \leq r$ with $x \in \mathbf{M}^{]0,s]}$. Choose a $\widetilde{\mathbf{B}}^{]0,r]}$-basis of $\mathbf{M}^{]0,r]}$ and write x with respect to this basis as a vector $v = (v_1, \ldots, v_n)^T \in \big(\widetilde{\mathbf{B}}^{]0,s]}\big)^{\oplus n}$. Let $A \in \mathrm{GL}_n\big(\widetilde{\mathbf{B}}^{]0,rp^{-a}]}\big)$ be the matrix by which the isomorphism $\varphi_{\mathbf{M}}^{]0,rp^{-a}]}$ acts on this basis. The equation $x = \varphi_{\mathbf{M}}(x)$ translates into $A^{-1}v = \big(\varphi^a(v_1), \ldots, \varphi^a(v_n)\big)^T$. Thus if $s \leq rp^{-a}$ we find $\varphi^a(v_i) \in \widetilde{\mathbf{B}}^{]0,s]}$, whence $v_i \in \widetilde{\mathbf{B}}^{]0,sp^a]}$. Continuing in this way we see that in fact $v_i \in \widetilde{\mathbf{B}}^{]0,r]}$, that is $x \in \mathbf{M}^{]0,r]}$. □

4 Constructing φ-Modules from Filtered Isocrystals

In the situation where the Hodge-Tate weights are 0 and 1 we will associate with any analytic point $\mu \in \breve{\mathcal{F}}^{\mathrm{an}}$ a φ-module $\mathbf{M}_\mu$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ where C is the completion of an algebraic closure of $\mathcal{H}(\mu)$. In case $\mathcal{H}(\mu)/\breve{E}$ finite this construction parallels Berger's construction [4, §II] that works for arbitrary Hodge-Tate weights and even produces an "$\mathcal{H}(\mu)$-rational" version of $\mathbf{M}_\mu$. We begin with the following

Proposition 4.1. *Let $\mathbf{N}$ be a φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ represented by a free $\widetilde{\mathbf{B}}^{]0,1]}$-module $\mathbf{N}^{]0,1]}$ and an isomorphism $\varphi_{\mathbf{N}}^{]0,p^{-1}]} : \mathbf{N}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\varphi} \widetilde{\mathbf{B}}^{]0,p^{-1}]} \xrightarrow{\ \sim\ } \mathbf{N}^{]0,p^{-1}]}$. Let $W_{\mathbf{M}}$ be a C-subspace of $W_{\mathbf{N}} := \mathbf{N}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$. Then there exists a uniquely determined φ-submodule $\mathbf{M} \subset \mathbf{N}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ with $t\mathbf{N} \subset \mathbf{M}$ which is represented by a $\widetilde{\mathbf{B}}^{]0,1]}$-submodule $t\mathbf{N}^{]0,1]} \subset \mathbf{M}^{]0,1]} \subset \mathbf{N}^{]0,1]}$ such that*

$$\mathbf{M}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C \;=\; W_{\mathbf{M}}$$

inside $W_{\mathbf{N}}$.

Proof. If I is a closed subinterval of $(0,1]$ set $\mathbf{N}^{I} := \mathbf{N}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]}} \widetilde{\mathbf{B}}^{I}$. Also if $I = [s,r]$ we let $pI := [ps, pr]$. The isomorphism $\varphi_{\mathbf{N}}^{]0,p^{-1}]}$ induces an isomorphism $\varphi_{\mathbf{N}}^{I} : \mathbf{N}^{pI} \otimes_{\widetilde{\mathbf{B}}^{pI},\varphi} \widetilde{\mathbf{B}}^{I} \xrightarrow{\ \sim\ } \mathbf{N}^{I}$. Let $I_n := [p^{-n-1}, p^{-n}]$ for $n \in \mathbb{N}_0$. We define $\mathbf{M}'^{I_0}$ as the preimage of $W_{\mathbf{M}}$ in $\mathbf{N}^{I_0}$ under the canonical morphism $\mathbf{N}^{I_0} \to \mathbf{N}^{i_0} \otimes_{\widetilde{\mathbf{B}}^{I_0},\theta} C = W_{\mathbf{N}}$ and we let $\mathbf{M}^{I_0}$ be the intersection of $\mathbf{M}'^{I_0}$ with $\varphi_{\mathbf{N}}^{[p^{-1},p^{-1}]}\big(\mathbf{M}'^{I_0} \otimes_{\widetilde{\mathbf{B}}^{I_0},\varphi} \widetilde{\mathbf{B}}^{[p^{-1},p^{-1}]}\big)$ inside $\mathbf{N}^{[p^{-1},p^{-1}]}$, that is $\mathbf{M}^{I_0}$ equals

$$\ker\Big(\ \mathbf{M}'^{I_0} \;\hookrightarrow\; \mathbf{N}^{[p^{-1},p^{-1}]} \;\twoheadrightarrow\; \mathbf{N}^{[p^{-1},p^{-1}]}/\varphi_{\mathbf{N}}^{[p^{-1},p^{-1}]}\big(\mathbf{M}'^{I_0} \otimes_{\widetilde{\mathbf{B}}^{I_0},\varphi} \widetilde{\mathbf{B}}^{[p^{-1},p^{-1}]}\big)\ \Big).$$

Since $\widetilde{\mathbf{B}}^{I_0}$ is a principal ideal domain by [24, Proposition 2.6.8] we see that $\mathbf{M}^{I_0}$ is a free $\widetilde{\mathbf{B}}^{I_0}$-submodule of $\mathbf{N}^{I_0}$ of full rank. For $n \geq 1$ we define $\mathbf{M}^{I_n}$ as the image of $\mathbf{M}^{I_0} \otimes_{\widetilde{\mathbf{B}}^{I_0},\varphi^n} \widetilde{\mathbf{B}}^{I_n}$ under the isomorphism $\varphi_{\mathbf{N}}^{I_n} \circ \ldots \circ \varphi_{\mathbf{N}}^{I_1} : \mathbf{N}^{I_0} \otimes_{\widetilde{\mathbf{B}}^{I_0},\varphi^n} \widetilde{\mathbf{B}}^{I_n} \xrightarrow{\ \sim\ } \mathbf{N}^{I_n}$. In the terminology of Kedlaya [24, §2.8] the collection $\mathbf{M}^{I_n}$ for $n \in \mathbb{N}_0$ defines a *vector bundle over* $\widetilde{\mathbf{B}}^{]0,1]}$ which by [24, Theorem 2.8.4] corresponds to a free $\widetilde{\mathbf{B}}^{]0,1]}$-submodule $\mathbf{M}^{]0,1]}$ of $\mathbf{N}^{]0,1]}$. By construction $t\mathbf{N}^{]0,1]} \subset \mathbf{M}^{]0,1]}$ and $\varphi_{\mathbf{N}}^{]0,p^{-1}]}$ restricts to an isomorphism on $\mathbf{M}^{]0,1]}$. This makes $\mathbf{M} := \mathbf{M}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]}} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ into a φ-module with the desired properties. Clearly $\mathbf{M}^{]0,1]}$ is uniquely determined by the subspace $W_{\mathbf{M}} \subset W_{\mathbf{N}}$ and the requirements that $t\mathbf{N}^{]0,1]} \subset \mathbf{M}^{]0,1]} \subset \mathbf{N}^{]0,1]}$ and $\varphi_{\mathbf{N}}^{I_n}\big(\mathbf{M}^{I_{n-1}} \otimes_{\widetilde{\mathbf{B}}^{I_{n-1}},\varphi} \widetilde{\mathbf{B}}^{I_n}\big) = \mathbf{M}^{I_n}$. $\square$

Now assume that K is a (not necessarily finite) extension of K_0. Let (D, φ_D) be an F-isocrystal over $\mathbb{F}_p^{\mathrm{alg}}$ and let $Fil^0 D_K$ be a K-subspace of $D_K = D \otimes_{K_0} K$. Let $Fil^{-1} D_K = D_K$ and $Fil^1 D_K = (0)$ and $\underline{D} = (D, \varphi_D, Fil^{\bullet} D_K)$. We set $\mathbf{D}^{]0,1]} := D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]}$ and $(\mathbf{D}, \varphi_{\mathbf{D}}) := (D, \varphi_D) \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$.

Definition 4.2. We let $\mathbf{M}(\underline{D})$ be the φ-submodule of $\mathbf{D}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ represented by the $\widetilde{\mathbf{B}}^{]0,1]}$-module $\mathbf{M}^{]0,1]}$ with $\mathbf{M}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C \;=\; (Fil^0 D_K) \otimes_K C$ inside $D_C = \mathbf{D}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$ whose existence was established in Proposition 4.1.

Unfortunately we do not know how to construct $\mathbf{M}(\underline{D})$ for filtered isocrystals $\underline{D}$ with Hodge-Tate weights other than 0 and 1. Therefore we cannot make $\underline{D} \mapsto \mathbf{M}(\underline{D})$ into a tensor functor and so we cannot use Berger's argument [4, Théorème IV.2.1] to prove the next theorem.

Theorem 4.3. $\deg \mathbf{M}(\underline{D}) \;=\; t_N(\underline{D}) - t_H(\underline{D})$.

Proof. By the Dieudonné-Manin classification [26] there exists an F-isocrystal $(D', \varphi_{D'})$ over $\mathbb{F}_p^{\mathrm{alg}}$ of rank one with $\varphi_{D'} = p^{t_N(\underline{D})} \cdot \varphi$ which is isomorphic to $\det(D, \varphi_D)$. So by construction $\deg \mathbf{D} = t_N(\underline{D})$. Moreover,

$$\dim_C(\mathbf{D}^{]0,1]}/\mathbf{M}^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C \;=\; \dim_C(D_K/Fil^0 D_K) \otimes_K C \;=\; -\,t_H(\underline{D})\,.$$

The theorem is thus a consequence of the following lemma. $\square$

Lemma 4.4. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be φ-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ represented by $\widetilde{\mathbf{B}}^{]0,1]}$-modules $\mathbf{M}_i^{]0,1]}$. Assume that $\mathbf{M}_1^{]0,1]} \supset \mathbf{M}_2^{]0,1]} \supset t\mathbf{M}_1^{]0,1]}$. Then*

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \;=\; \dim_C (\mathbf{M}_1^{]0,1]}/\mathbf{M}_2^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C\,.$$

Proof. Clearly the equality holds for $\mathbf{M}_2 = t\mathbf{M}_1 \cong \mathbf{M}_1 \otimes \mathbf{M}(1,1)$ since $\deg t\mathbf{M}_1 - \deg \mathbf{M}_1 = \mathrm{rk}\, \mathbf{M}_1$. We claim that it suffices to prove the inequality

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \;\geq\; \dim_C W \tag{4.1}$$

where we abbreviate $W := (\mathbf{M}_1^{]0,1]}/\mathbf{M}_2^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$. Indeed we apply the inequality to the two inclusions $\mathbf{M}_1 \supset \mathbf{M}_2 \supset t\mathbf{M}_1$ and $\mathbf{M}_2 \supset t\mathbf{M}_1 \supset t\mathbf{M}_2$ and conclude using the exact sequence of $\widetilde{\mathbf{B}}^{]0,1]}$-modules

$$0 \longrightarrow \mathbf{M}_2^{]0,1]}/t\mathbf{M}_1^{]0,1]} \longrightarrow \mathbf{M}_1^{]0,1]}/t\mathbf{M}_1^{]0,1]} \longrightarrow \mathbf{M}_1^{]0,1]}/\mathbf{M}_2^{]0,1]} \longrightarrow 0\,.$$

To prove the inequality (4.1) we argue by induction on $\dim_C W$. Let $\dim_C W = 1$. Since $\det \mathbf{M}_1 \supset \det \mathbf{M}_2$ we know that $\deg \mathbf{M}_2 \geq \deg \mathbf{M}_1$ from Proposition 3.5. If we had $\deg \mathbf{M}_2 = \deg \mathbf{M}_1$ then $\mathbf{M}_2 = \mathbf{M}_1$ by [24, Lemma 3.4.2]. So $\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \geq 1 = \dim_C W$ as desired.

Let now $\dim_C W > 1$ and choose a C-subspace W' of dimension 1 of W. By Proposition 4.1 there is a unique $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-submodule $\mathbf{M}_2 \subset \mathbf{M}_3 \subset \mathbf{M}_1$ corresponding to W'. By induction $\deg \mathbf{M}_3 - \deg \mathbf{M}_1 \geq \dim_C (W/W')$ and $\deg \mathbf{M}_2 - \deg \mathbf{M}_3 \geq \dim_C W'$ proving the lemma. $\square$

If K/K_0 is finite one can check that $\mathbf{M}(\underline{D})$ equals the φ-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ constructed by Berger [4, §II]. One of Berger's main theorems is the following criterion.

Theorem 4.5. *([4, §IV.2]) Let K be a finite extension of K_0. Then $\underline{D}$ is admissible if and only if $\mathbf{M}(\underline{D}) \cong \mathbf{M}(0,1)^{\oplus \dim D}$.*

Let us make explicit what happens otherwise.

Proposition 4.6. *Assume that $t_N(\underline{D}) = t_H(\underline{D})$. Then $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0,1)^{\oplus \dim D}$ if and only if for some (any) integer $e \geq (\dim D) - 1$ there exists a non-zero $x \in \mathrm{H}^0_{\varphi^e}\big([e]_* \mathbf{D} \otimes \mathbf{M}(1,1)\big)$ with $\theta\big(\varphi^m_{\mathbf{D}}(x)\big) \in (Fil^0 D_K) \otimes_K C$ for all $m = 0, \ldots, e-1$.*

Note that $\mathbf{D}$ can be represented by $(D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,r]}, b\cdot\varphi)$ for arbitrarily large $r > 0$. Hence by Proposition 3.10 the element x actually belongs to $D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,r]}$ and the expression $\theta\big(\varphi^m_{\mathbf{D}}(x)\big)$ makes sense.

Proof. Let $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0,1)^{\oplus \dim D}$ and let $e \geq (\dim D) - 1$ be any integer. By Theorems 4.3 and 3.7 there is a φ-module $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ with $c < 0$ and $d < \dim D$ which is a summand of $\mathbf{M}(\underline{D})$. Thus by Proposition 3.5 there exists a non-zero morphism of φ-modules $f : \mathbf{M}(-1,e) \to \mathbf{M}(c,d) \subset \mathbf{M}(\underline{D}) \subset \mathbf{D}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_e$ be a basis of $\mathbf{M}(-1,e)$ satisfying (3.1) and let x be the image of $\mathbf{e}_1$ in $\mathbf{D}$ under f. Then $\varphi^e_{\mathbf{D}}(x) = p^{-1}x$, that is $x \in \mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D} \otimes \mathbf{M}(1,1)\big)$. Moreover, $f(\mathbf{e}_{m+1}) = \varphi^m_{\mathbf{D}}(x)$ in $\mathbf{D}$ for $0 \leq m < e$. Now the fact that the morphism f factors through $\mathbf{M}(\underline{D})$ amounts by Proposition 4.1 to $\theta\big(\varphi^m_{\mathbf{D}}(x)\big) \in (Fil^0 D_K) \otimes_K C$.

Conversely assume that for some integer $e \geq (\dim D) - 1$ there exists a non-zero element x in $\mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D} \otimes \mathbf{M}(1,1)\big)$ with $\theta\big(\varphi^m_{\mathbf{D}}(x)\big) \in (Fil^0 D_K) \otimes_K C$ for all $m = 0, \ldots, e-1$. Define the non-trivial morphism of φ-modules $f : \mathbf{M}(-1,e) \to \mathbf{D}$ by $f(\mathbf{e}_{m+1}) := \varphi^m_{\mathbf{D}}(x)$ for $0 \leq m < e$. Since $\theta\big(\varphi^m_{\mathbf{D}}(x)\big) \in (Fil^0 D_K) \otimes_K C$ the morphism f factors through $\mathbf{M}(\underline{D})$ by Proposition 4.1. By Proposition 3.5 this implies $\mathbf{M}(\underline{D}) \ncong \mathbf{M}(0,1)^{\oplus \dim D}$. $\square$

5 The Minuscule Case

We retain the notation from Section 1. In particular G is a reductive group and $\{\mu\}$ is a conjugacy class of cocharacters of G. We make the following assumption on the pair $(G, \{\mu\})$:

$$\begin{array}{c}\text{There exists a faithful } \mathbb{Q}_p\text{-rational representation } V \text{ of } G \\ \text{such that all the weights of } \{\mu\} \text{ on } V \text{ are 0 or } -1.\end{array} \tag{5.1}$$

If moreover G possesses no proper normal subgroup defined over $\mathbb{Q}_p$ containing the image of $\{\mu\}$, Serre [30, Lemma 3.2.5] has shown that for every simple factor of G there is an element μ in $\{\mu\}$ which is a minuscule coweight of that factor. Moreover any simple factor of G is of type A, B, C or D. Conversely if G is simple and of type A, B, C or D and μ is a minuscule coweight of G then condition (5.1) is satisfied. Namely one can take V to be the contragredient of the standard representation (A), the spin representation (B), the standard representation (C), or respectively the standard representation or a half-spin representation (D).

Assume now that $(G, \{\mu\})$ satisfies condition (5.1) and let $G' = \mathrm{GL}(V)$. Then G is a closed subgroup of G'. This defines a closed embedding $\mathcal{F} \hookrightarrow \mathcal{F}' := \mathcal{F}lag(V)$ into the flag variety from (1.2). Here $\mathcal{F}lag(V)$ is actually a Grassmannian. Let $b \in G(K_0)$ and view it as an element of $G'(K_0)$. We obtain a closed embedding of $\breve{E}$-analytic period spaces $\breve{\mathcal{F}}^{wa}_b \hookrightarrow \breve{\mathcal{F}}'^{wa}_b$. We denote by $\breve{\mathcal{F}}^{\mathrm{an}}$ the $\breve{E}$-analytic space associated with $\mathcal{F} \otimes_E \breve{E}$; see Appendix A.

Let $\mu \in \breve{\mathcal{F}}^{\mathrm{an}}$ be an analytic point. Let $K = \mathcal{H}(\mu)$ and let C be the completion of an algebraic closure of K. Let $\underline{D}_\mu := (V_{K_0}, b \cdot \varphi, Fil^\bullet_\mu D_K)$. In particular $Fil^{-1} D_K = D_K$ and $Fil^1 D_K = (0)$. So we let $\mathbf{M}_\mu := \mathbf{M}(\underline{D}_\mu)$ be the φ-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ from Definition 4.2. We define

$$\breve{\mathcal{F}}^a_b \; := \; \big\{ \mu \in \breve{\mathcal{F}}^{\mathrm{an}} \text{ analytic points} : \; \mathbf{M}_\mu \cong \mathbf{M}(0,1)^{\oplus d} \big\} . \tag{5.2}$$

If $b' = g\, b\, \varphi(g^{-1})$ for some $g \in G(K_0)$ one easily checks that $\mu \mapsto g^{-1}\mu g$ maps $\breve{\mathcal{F}}^a_b$ isomorphically onto $\breve{\mathcal{F}}^a_{b'}$. If $t_N(\underline{D}_\mu) \neq t_H(\underline{D}_\mu) = \dim V - \dim_K Fil^0_\mu D_K$ the sets $\breve{\mathcal{F}}^{wa}_b$ and $\breve{\mathcal{F}}^a_b$ are empty. So from now on we assume $t_N(\underline{D}_\mu) = t_H(\underline{D}_\mu)$ for all $\mu \in \breve{\mathcal{F}}^{\mathrm{an}}$.

Proposition 5.1. *The set $\breve{\mathcal{F}}^a_b$ is contained in $\breve{\mathcal{F}}^{wa}_b$.*

Proof. Let $\mu \in \breve{\mathcal{F}}^a_b$ be an analytic point and set $K = \mathcal{H}(\mu)$. Let $D' \subset D$ be a φ_D-stable K_0-subspace and let $Fil^i_\mu D'_K := D'_K \cap Fil^i_\mu D_K$. We have to show that $t_H(\underline{D}') \leq t_N(\underline{D}')$ for the subobject $\underline{D}' = (D', \varphi_D|_{D'}, Fil^\bullet_\mu D'_K) \subset \underline{D}_\mu$ with equality if $\underline{D}' = \underline{D}_\mu$. Consider the φ-submodule $\mathbf{M}' := \mathbf{M}(\underline{D}') \subset \mathbf{M}(\underline{D}_\mu)$. Then $\mathrm{rk}\, \mathbf{M}' \cdot \mathrm{wt}\, \mathbf{M}' = \deg \mathbf{M}' = t_N(\underline{D}') - t_H(\underline{D}')$ by Theorem 4.3. Since $\mu \in \breve{\mathcal{F}}^a_b$ we have $\mathbf{M}(\underline{D}_\mu) \cong \mathbf{M}(0,1)^{\oplus \dim D}$ and hence $t_H(\underline{D}_\mu) = t_N(\underline{D}_\mu)$. Moreover, $\mathrm{wt}\, \mathbf{M}' \geq \mathrm{wt}\, \mathbf{M}(\underline{D}_\mu) = 0$ by Proposition 3.9 proving our claim. $\square$

Theorem 5.2. *The set $\breve{\mathcal{F}}_b^a$ is an open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{\mathrm{an}}$. If b is decent with the integer s then $\breve{\mathcal{F}}_b^a$ has a natural structure of open E_s-analytic subspace of $(\mathcal{F}\otimes_E E_s)^{\mathrm{an}}$ from which it arises by base change to $\breve{E}$.*

We immediately deduce the following

Corollary 5.3. *The open immersion $\breve{\mathcal{F}}_b^a \subset \breve{\mathcal{F}}_b^{wa}$ induces an étale morphism of rigid analytic spaces $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ which is bijective on rigid analytic points. It is an isomorphism if and only if $\breve{\mathcal{F}}_b^a = \breve{\mathcal{F}}_b^{wa}$.*

Proof. The functor $(\ )^{\mathrm{rig}}$ takes étale morphisms to étale morphisms. The fact that the morphism of rigid analytic spaces is bijective on rigid analytic points (the ones with residue field finite over $\breve{E}$) follows from the Colmez-Fontaine Theorem [12]. The rest is a consequence of Theorem A.2 since $\breve{\mathcal{F}}_b^a$ and $\breve{\mathcal{F}}_b^{wa}$ are paracompact by Lemma A.3. $\square$

Proof of Theorem 5.2. (a) Consider the faithful representation V from (5.1) and let $G' = \mathrm{GL}(V)$. This defines a closed embedding $\mathcal{F} \hookrightarrow \mathcal{F}' := \mathcal{F}lag(V)\otimes_{\mathbb{Q}_p} E$ into the flag variety from (1.2). Here $\mathcal{F}lag(V)$ is a Grassmannian isomorphic to G'/S' where $S' = \mathrm{Stab}_{G'}(V_0)$ is the stabilizer of an appropriate subspace V_0 of V. By definition $\breve{\mathcal{F}}_b^a = \breve{\mathcal{F}}^{\mathrm{an}} \cap \breve{\mathcal{F}}_b'^{a}$. So it suffices to prove the theorem for G' instead of G. Since G' is connected we may assume by [25] that b is decent, say with integer s. We let $\mathcal{F}_s'^{\mathrm{an}} := (\mathcal{F}'\otimes_E E_s)^{\mathrm{an}}$ and define the subset $\mathcal{F}_b'^{a} \subset \mathcal{F}_s'^{\mathrm{an}}$ by the same condition as in (5.2). We show that it is open. Choose an integer $e \geq (\dim V) - 1$ which is a multiple of s. Then by Proposition 4.6 the set $\mathcal{F}_s'^{\mathrm{an}} \smallsetminus \mathcal{F}_b'^{a}$ equals

$$\Big\{ \mu \in \mathcal{F}_s^{\mathrm{an}} \text{ analytic points : there exists an algebraically closed complete extension } C \text{ of } \mathcal{H}(\mu) \text{ and an element } x \in \mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D}_C \otimes \mathbf{M}(1,1)\big),\ x \neq 0 \text{ with } \theta\big(\varphi_{\mathbf{D}}^m(x)\big) \in (Fil_\mu^0 V_{\mathcal{H}(\mu)}) \otimes_{\mathcal{H}(\mu)} C \text{ for all } m = 0,\ldots,e-1 \Big\}.$$

Here $[e]_*\mathbf{D}_C$ is the φ^e-module $(D,\varphi_d^e)\otimes_{K_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$.

(b) We identify $V\otimes_{\mathbb{Q}_p}\mathbb{A}^1_{E_s}$ with affine d-space $\mathbb{A}^d_{E_s}$ over E_s. For $\eta \in E_s$ consider the $\breve{E}$-analytic polydisc $\mathbb{D}(\eta)^{de} = \mathcal{M}\big(E_s\langle \frac{h_{im}}{\eta} : i = 1,\ldots,d;\, m = 0,\ldots,e-1\rangle\big) \subset (\mathbb{A}^d_{E_s})^e$ with radii $(|\eta|,\ldots,|\eta|)$. We will construct in (c) below a constant $\eta \in E_s$ and a compact subset Z of $\mathbb{D}(\eta)^{de}$ with the following property: If C is an algebraically closed complete extension of E_s and $x \in \mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D}_C\otimes\mathbf{M}(1,1)\big)$ with $x\neq 0$ then for some integer N

$$\big(h_m\big)_{m=0}^{e-1} \;:=\; \Big(\big(h_{1m},\ldots,h_{dm}\big)^T\Big)_{m=0}^{e-1} \;:=\; \Big(p^N\theta\big(\varphi_{\mathbf{D}}^m(x)\big)\Big)_{m=0}^{e-1}$$

is a C valued point of Z and Z consists precisely of those points.

Now let $G_s'^{\mathrm{an}}$ be the E_s-analytic space associated with the group scheme $G'\otimes_{\mathbb{Q}_p} E_s$ and consider the morphism of E_s-analytic spaces

$$\begin{aligned} \beta:\ G_s'^{\mathrm{an}}\times_{E_s}\mathbb{D}(\eta)^{de} &\longrightarrow (\mathbb{A}^d_{E_s})^e \;\cong\; (V\otimes_{\mathbb{Q}_p}\mathbb{A}^1_{E_s})^e \\ \big(g\ ,\ (h_m)_{m=0}^{e-1}\big) &\longmapsto \big(g^{-1}h_m\big)_{m=0}^{e-1}\,. \end{aligned}$$

Let Y be the closed subset of $G_s'^{\mathrm{an}}\times_{E_s}\mathbb{D}(\eta)^{de}$ defined by the condition that $(h_m)_{m=0}^{e-1}$ belongs to Z and that $\beta(Y) \subset (V_0\otimes_{\mathbb{Q}_p}\mathbb{A}^1_{E_s})^e$. Furthermore consider the projection map

$$pr_1:\ G_s'^{\mathrm{an}}\times_{E_s}\mathbb{D}(\eta)^{de} \longrightarrow G_s'^{\mathrm{an}}$$

onto the first factor and the canonical map $\gamma : {G'_s}^{\mathrm{an}} \to {\mathcal{F}'_s}^{\mathrm{an}}$ coming from the isomorphism $\mathcal{F}'_s \cong G'_s/\operatorname{Stab}_{G'_s}(V_0)$. Then $\mu \in {\mathcal{F}'_s}^{\mathrm{an}}$ does not belong to ${\mathcal{F}'_b}^a$ if and only if $\mu \in \gamma \circ pr_1(Y)$. Since $\mathbb{D}(\eta)^{de}$ is quasi-compact the projection pr_1 is a proper map of topological Hausdorff spaces by [5, Proposition 3.3.2]. Thus in particular it is closed and $pr_1(Y)$ is closed. Note that ${\mathcal{F}'_s}^{\mathrm{an}}$ carries the quotient topology under γ since γ is a smooth morphism of schemes, hence open by [6, Proposition 3.5.8 and Corollary 3.7.4]. Since by construction $pr_1(Y) = \gamma^{-1}\big(\gamma \circ pr_1(Y)\big)$ we conclude that ${\mathcal{F}'_b}^a = {\mathcal{F}'_s}^{\mathrm{an}} \smallsetminus \gamma \circ pr_1(Y)$ is open in ${\mathcal{F}'_s}^{\mathrm{an}}$ as desired.

(c) It remains to construct the compact set Z. Since b is decent, the φ-module $[e]_*\mathbf{D} \otimes \mathbf{M}(1,1)$ is isomorphic to $\bigoplus_{i=1}^d \mathbf{M}(-c_i, 1)$ for suitable integers c_i. We assume that the identification of $V \otimes_{\mathbb{Q}_p} \mathbb{A}^1_{E_s}$ with $\mathbb{A}^d_{E_s}$ in (b) was chosen compatible with this direct sum decomposition. Let $c_1, \ldots, c_k > 0 = c_{k+1} = \ldots = c_\ell > c_{\ell+1}, \ldots, c_d$. Then by Proposition 3.6

$$
\begin{aligned}
\mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D} \otimes \mathbf{M}(1,1)\big) \quad &\cong \quad \bigoplus_{i=1}^{k} \Big\{ \sum_{\nu\in\mathbb{Z}} p^{c_i\nu} \sum_{j=0}^{c_i-1} p^j \varphi^{-e\nu}([u_{ij}]) : \ u_{ij} \in \widetilde{\mathbf{E}},\, v_{\mathbf{E}}(u_{ij}) > 0 \Big\} \ \oplus \\
&\oplus \quad \bigoplus_{i=k+1}^{\ell} W(\mathbb{F}_{p^e})[\tfrac{1}{p}] \ \oplus \\
&\oplus \quad \bigoplus_{i=\ell+1}^{d} (0) \quad .
\end{aligned}
$$

For $1 \le i \le k, 0 \le j \le c_i - 1$ consider the compact sets

$$
\begin{aligned}
U_{ij}^{(0)} \quad &:= \quad \mathcal{M}\big(E_s\langle \tfrac{u_{ij}^{(0)}}{p} \rangle\big) \quad = \quad \big\{\, |u_{ij}^{(0)}| \le |p| \,\big\} \qquad \text{and} \\
U_{ij}^{(n)} \quad &:= \quad \mathcal{M}\big(E_s\langle u_{ij}^{(n)} \rangle\big) \quad = \quad \big\{\, |u_{ij}^{(n)}| \le 1 \,\big\} .
\end{aligned}
$$

Then the sets

$$
U_{ij} \quad := \quad \Big\{ (u_{ij}^{(n)})_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} U_{ij}^{(n)} : \ (u_{ij}^{(n+1)})^p = u_{ij}^{(n)} \text{ for all } n \ge 0 \Big\} \quad \text{and}
$$

$$
U \quad := \quad \prod_{i=1}^{k} \prod_{j=0}^{c_i-1} U_{ij} \times \prod_{i=k+1}^{\ell} W(\mathbb{F}_{p^e}) \times \prod_{i=\ell+1}^{d} \{0\}
$$

are compact by Tychonoff's theorem. For an arbitrary algebraically closed extension C of E_s consider a C-valued point u of U given by

$$
\Big(\big((u_{ij}^{(n)})_{n\in\mathbb{N}_0}\big)_{i=1,\ldots,k\,;\, j=0,\ldots,c_i-1} ,\ (a_i)_{i=k+1,\ldots,\ell} ,\ (0)_{i=\ell+1,\ldots,d} \Big)
$$

with $u_{ij}^{(n)} \in C$ and $a_i \in W(\mathbb{F}_{p^e})$. We assign to u the C-valued point y of $\mathbb{A}^{de}_{E_s}$ with $(h_m)_{m=0}^{e-1} = \big(\theta(\varphi^m_{\mathbf{D}}(x))\big)_{m=0}^{e-1}$ where x is the element of $\mathrm{H}^0_{\varphi^e}\big([e]_*\mathbf{D} \otimes \mathbf{M}(1,1)\big)$ associated with the $u_{ij}^{(n)}$ and a_i. In concrete terms this means

$$
h_m \quad := \quad b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \begin{pmatrix} \Big(\sum_{j=0}^{c_i-1} p^j \Big(\sum_{\nu>0} p^{c_i\nu} u_{ij}^{(e\nu-m)} \ + \ \sum_{\nu\le 0} p^{c_i\nu} \big(u_{ij}^{(0)}\big)^{p^{m-\nu e}} \Big) \Big)_{i=1}^{k} \\ \big(\varphi^m(a_i)\big)_{i=k+1}^{\ell} \\ (0)_{i=\ell+1}^{d} \end{pmatrix}
$$

for $m = 0, \dots, e-1$. This defines a map $\alpha : U \to \mathbb{A}^{de}_{E_s}, u \mapsto y$ of topological Hausdorff spaces. We claim that α is continuous. By definition of the topology on $\mathbb{A}^{de}_{E_s}$ (see Appendix A)the map α is continuous if and only if for any polynomial $f \in E_s[h_{im} : i = 1, \dots, d,\, m = 0, \dots, e-1]$ and any open interval $I \subset \mathbb{R}$ the preimage under α of the open set

$$W \;=\; \left\{ y \in \mathbb{A}^{de}_{E_s} :\; |f|_y \in I \right\}$$

is open in U. Since $|u^{(0)}_{ij}|_u \le |p|$ there is a constant $\eta \in E_s$ such that $|h_{im}|_{\alpha(u)} \le |\eta|$ for all i and m and all $u \in U$. Write $f = \sum_{\underline{n}} b_{\underline{n}} \underline{h}^{\underline{n}}$ where $b_{\underline{n}} \in E_s$ for every multi index $\underline{n} \in \mathbb{N}_0^{de}$. If we set $h_{im} = h'_{im} + h''_{im}$ we obtain from the Taylor expansion of f in powers of $\underline{h}''$ a bound δ such that the condition $|h''_{im}|_y \le \delta$ for all i, m implies

$$|f(\underline{h})|_y \in I \text{ and } |\underline{h}|_y \le |\eta| \qquad \Longleftrightarrow \qquad |f(\underline{h}')|_y \in I \text{ and } |\underline{h}'|_y \le |\eta| \,.$$

Now we fix a positive integer r and set for all $m = 0, \dots, e-1$

$$\big(h''_{im}\big)_{i=1}^{d} \;:=\; b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \begin{pmatrix} \Big(\sum\limits_{j=0}^{c_i-1} p^j \sum\limits_{\nu > r} p^{c_i \nu} u^{(e\nu - m)}_{ij}\Big)_{i=1}^{k} \\ (0)_{i=k+1}^{\ell} \\ (0)_{i=\ell+1}^{d} \end{pmatrix} \qquad \text{and}$$

$$\big(h'_{im}\big)_{i=1}^{d} \;:=\; b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \begin{pmatrix} \Big(\sum\limits_{j=0}^{c_i-1} p^j \sum\limits_{\nu \le r} p^{c_i \nu} \big(u^{(er)}_{ij}\big)^{p^{m+(r-\nu)e}}\Big)_{i=1}^{k} \\ \big(\varphi^m(a_i)\big)_{i=k+1}^{\ell} \\ (0)_{i=\ell+1}^{d} \end{pmatrix} .$$

Since $|u^{(n)}_{ij}|_u < 1$ for any n, i, j and for any point $u \in U$ we can find a large enough integer r such that $|h''_{im}|_{\alpha(u)} \le \delta$ for all i, m and for all $u \in U$. This shows that

$$\alpha^{-1}(W) \;=\; W' \;:=\; \left\{ u \in U :\; |f(h'_{im})|_{\alpha(u)} \in I \right\}.$$

To prove that W' is open in U observe that the map

$$\begin{aligned} \alpha'_m :\; U' \;:=\; \prod_{i=1}^{k} \prod_{j=0}^{c_i-1} U^{(er)}_{ij} \times \prod_{i=k+1}^{\ell} \mathbb{A}^1_{E_s} \;&\longrightarrow\; \mathbb{A}^d_{E_s} \\ \Big((u^{(er)}_{ij})_{i,j} \,,\; \big(\varphi^m(a_i)\big)_i \Big) \;&\longmapsto\; (h'_{im})_{i=1}^{d} \end{aligned}$$

is a morphism of E_s-analytic spaces, and in particular continuous. Furthermore, the projection maps $\alpha_{ij} : U_{ij} \to U^{(er)}_{ij}$ and the inclusion

$$\alpha_i :\; W(\mathbb{F}_{p^s}) \;\longrightarrow\; \mathbb{A}^e_{E_s} \;=\; \mathcal{M}\big(E_s[T_0, \dots, T_{e-1}]\big) \,, \quad a \;\longmapsto\; \big(T_m \mapsto \varphi^m(a)\big)$$

for $i = k+1, \dots, \ell$ are continuous. Since

$$\alpha' \;:=\; \Big(\prod_{i=1}^{k} \prod_{j=0}^{c_i-1} \alpha_{ij} \times \prod_{i=k+1}^{\ell} \alpha_i\Big)^{-1} (\alpha'_0 \times \ldots \times \alpha'_{e-1})^{-1} :\; U \;\longrightarrow\; \mathbb{A}^{de}_{E_s}$$

maps u to $\alpha'(u) = (h'_m)_{m=0}^{e-1}$ the set $W' = (\alpha')^{-1}(W)$ is open in U and this proves that α is continuous.

Now multiplying $(h_{im})_{i,m}$ with p amounts to replacing $u_{ij}^{(n)}$ by $u_{i,j-1}^{(n)}$ for $j = 1, \ldots, c_i - 1$, and $u_{i,0}^{(n)}$ by $(u_{i,c_i-1}^{(n)})^{p^e}$, and a_i by pa_i. Thus we may take

$$Z \;:=\; \alpha\Big(U \smallsetminus \big\{\, u \in U :\ |u_{ij}^{(0)}|_u < |p|^{p^e},\ a_i \in pW(\mathbb{F}_{p^e}) \text{ for all } i \text{ and } j \,\big\}\Big)\,.$$

Then Z is the continuous image of a compact set and satisfies the property required in (b). This proves the theorem. $\square$

We give an example showing that the inclusion $\mathcal{F}_b^a \subset \mathcal{F}_b^{wa}$ may be strict. Similar examples were independently obtained by V. Lafforgue.

Example 5.4. Let $G = \mathrm{GL}_5(\mathbb{Q}_p)$ and consider the conjugacy class $\{\mu\}$ of cocharacters containing $\mu : \mathbb{G}_m \to G$, $z \mapsto \mathrm{diag}(z^{-1}, z^{-1}, z^{-1}, 1, 1)$. We have $E = \mathbb{Q}_p$ and $\mathcal{F} \cong \mathrm{Grass}(2,5)$ the Grassmannian. Let

$$b \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & p^{-3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \;\in\; G(K_0)\,.$$

The element b is decent with integer $s = 5$. We take $E_s = W(\mathbb{F}_{p^5})[\frac{1}{p}]$. Since the isocrystal $(V_{K_0}, b\cdot\varphi)$ is simple every cocharacter in $\{\mu\}$ is weakly admissible, that is $\mathcal{F}_b^{wa} = \mathcal{F}_s^{\mathrm{an}}$.

Let $\mu \in \mathcal{F}_s^{\mathrm{an}}$ and let C be the completion of an algebraic closure of $\mathcal{H}(\mu)$. The φ-module $\mathbf{M} = \mathbf{M}(V_{K_0}, b\cdot\varphi, Fil_\mu^\bullet V_{\mathcal{H}(\mu)})$ constructed from b and μ satisfies $\deg \mathbf{M} = 0$ and

$$\mathbf{D} \;=\; \mathbf{M}(-3,5) \;\supset\; \mathbf{M} \;\supset\; t\mathbf{D} \;=\; \mathbf{D} \otimes \mathbf{M}(1,1) \;=\; \mathbf{M}(2,5)$$

where $\mathbf{D} = (V_{K_0}, b\cdot\varphi) \otimes_{K_0} \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$. Since by Proposition 3.5 the weight of every summand of $\mathbf{M}$ lies between $-3/5$ and $2/5$, either $\mathbf{M} \cong \mathbf{M}(0,1)^{\oplus 5}$ or $\mathbf{M} \cong \mathbf{M}(-1,2) \oplus \mathbf{M}(1,3)$. (The first entries must sum to $0 = \deg \mathbf{M}$ and the second entries must sum to $5 = \mathrm{rk}\, \mathbf{M}$.) The bad situation $\mathbf{M} \cong \mathbf{M}(-1,2) \oplus \mathbf{M}(1,3)$ occurs if and only if

$$(0) \;\neq\; \mathrm{Hom}_\varphi\big(\mathbf{M}(-1,2), \mathbf{M}\big) \;=\; \Big\{\, f \in \mathrm{Hom}_\varphi\big(\mathbf{M}(-1,2), \mathbf{D}\big) :\ \theta(\mathrm{im}\, f) \subset Fil_\mu^0 V_{\mathcal{H}(\mu)} \,\Big\}\,.$$

One easily checks that $\mathrm{Hom}_\varphi\big(\mathbf{M}(-1,2), \mathbf{M}(-3,5)\big) =$

$$= \left\{\, A \;=\; \begin{pmatrix} \varphi^5(x) & x \\ \varphi^{11}(x) & \varphi^6(x) \\ \varphi^{17}(x) & \varphi^{12}(x) \\ \varphi^{23}(x) & \varphi^{18}(x) \\ \varphi^{29}(x) & \varphi^{24}(x) \end{pmatrix} \;:\; \begin{array}{c} x \;=\; \displaystyle\sum_{\nu \in \mathbb{Z}} p^\nu \varphi^{-10\nu}([u]) \;\in\; \mathrm{H}^0_{\varphi^{10}}\big(\mathbf{M}(1,1)\big) \\ u \;\in\; \mathbf{E}(C),\ 0 < v_{\mathbf{E}}(u) < \infty \end{array} \,\right\}.$$

Thus the points $\mu \in \mathcal{F}_s^{\mathrm{an}}$ for which the columns of $\theta(A)$ span $Fil_\mu^0 V_{\mathcal{H}(\mu)}$ do not belong to $\mathcal{F}_b^a$ and such points exist! This proves that the inclusion $\mathcal{F}_b^a \subset \mathcal{F}_b^{wa}$ is strict.

Remark 5.5. The example can easily be adapted to the general situation showing that only in rare cases $\mathcal{F}_b^a$ will equal $\mathcal{F}_b^{wa}$. Moreover, the question whether the two sets are equal depends largely on the combinatorics of the weights.

6 Relation with Period Morphisms

Rapoport and Zink also study a period morphism $\pi : \mathcal{M}^{\mathrm{an}} \to \breve{\mathcal{F}}_b^{wa}$. This is constructed as follows. Let $G = \mathrm{GL}(V)$ for a finite dimensional $\mathbb{Q}_p$-vector space V. Let $\breve{\mathcal{F}}$ be the Grassmanian over K_0 of d-dimensional subspaces of V_{K_0}. Let $b \in G(K_0)$ and $(D, \varphi_D) = (V_{K_0}, b\cdot\varphi)$. Assume that there exists a p-divisible group $\mathbf{X}$ over $\mathbb{F}_p^{\mathrm{alg}}$ of dimension d whose covariant Dieudonné isocrystal is (D, φ_D); see Messing [27]. Thus $t_N(D, \varphi_D) = \dim V - d$. Let $\mathcal{N}ilp_W$ be the category of $W := W(\mathbb{F}_p^{\mathrm{alg}})$-schemes on which p is locally nilpotent. For $S \in \mathcal{N}ilp_W$ denote by $\bar{S}$ the closed subscheme defined by the ideal $p\mathcal{O}_S$. The contravariant functor $\mathcal{N}ilp_W \longrightarrow \mathcal{S}ets$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of } (X, \rho : \mathbf{X}_{\bar{S}} \to X_{\bar{S}}) \text{ where} \\ \quad X \text{ is a } p\text{-divisible group over } S \text{ and} \\ \quad \rho \text{ is a quasi-isogeny over } \bar{S} \end{array} \right\}$$

is representable by a formal scheme $\mathcal{M}$ locally formally of finite type over $\mathrm{Spf}\, W$ by [28, Theorem 2.16]. Rapoport and Zink also study formal moduli schemes corresponding to additional data on (X, ρ) of type EL and PEL. Their period morphisms are derived from the following prototype.

Let $(X, \rho : \mathbf{X}_{\overline{\mathcal{M}}} \to X_{\overline{\mathcal{M}}})$ be the universal p-divisible group with quasi-isogeny over $\mathcal{M}$. Let $\mathbb{D}(X)_{\mathcal{M}}$ be the Lie algebra of the universal vector extension of X over $\mathcal{M}$ (see Messing [27]) and let $\mathcal{M}^{\mathrm{an}}$ be the K_0-analytic space associated with $\mathcal{M}$; see Theorem A.2. The quasi-isogeny ρ induces an isomorphism

$$D \otimes_{K_0} \mathcal{O}_{\mathcal{M}^{\mathrm{an}}} \xrightarrow{\ \sim\ } \mathbb{D}(X)_{\mathcal{M}^{\mathrm{an}}}$$

see [28, Proposition 5.15]. The kernel of the morphism

$$V_{K_0} \otimes_{K_0} \mathcal{O}_{\mathcal{M}^{\mathrm{an}}} \xrightarrow{\ \sim\ } \mathbb{D}(X)_{\mathcal{M}^{\mathrm{an}}} \longrightarrow (\mathrm{Lie}\, X)_{\mathcal{M}^{\mathrm{an}}}$$

defines an $\mathcal{M}^{\mathrm{an}}$-valued point of the Grassmannian $\breve{\mathcal{F}}^{\mathrm{an}}$. This is the desired *period morphism* $\pi : \mathcal{M}^{\mathrm{an}} \to \breve{\mathcal{F}}^{\mathrm{an}}$. It factors through $\breve{\mathcal{F}}_b^{wa}$.

Proposition 6.1. *The period morphism factors through $\breve{\mathcal{F}}_b^{a}$.*

Proof. Let $x \in \mathcal{M}^{\mathrm{an}}$ be an analytic point and let $\mu = \pi(x) \in \breve{\mathcal{F}}^{\mathrm{an}}$. Let $K = \mathcal{H}(x)$ and let C be the completion of an algebraic closure of K. Let X_x be the fiber of the universal p-divisible group X at x and consider the Tate module T_pX_x of X_x. An element $\lambda \in T_pX_x$ corresponds to a morphism of p-divisible groups $\lambda : \mathbb{Q}_p/\mathbb{Z}_p \to X_{\mathcal{O}_C}$ over $\mathcal{O}_C$. By functoriality of the universal vector extension this yields the following diagram of C-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C \arrow[r, equal] \arrow[d] & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C \arrow[r] \arrow[d, "\mathbb{D}(\lambda)_C"] & 0 & \\ 0 \arrow[r] & (Fil_\mu^0 D_K) \otimes_K C \arrow[r] & D \otimes_{K_0} C \arrow[r] & (\mathrm{Lie}\, X_x)_C \arrow[r] & 0\,. \end{tikzcd} \tag{6.1}$$

Note that $\mathrm{Lie}\, \mathbb{Q}_p/\mathbb{Z}_p = (0)$ and that $Fil_\mu^0 D_K$ is the K-subspace of D_K associated with X_x via the period morphism. We also obtain a morphism of crystals $\mathbb{D}(\lambda) : \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{D}(X_{\mathcal{O}_C})$ which we evaluate on the pd-thickening $\mathbf{B}_{\mathrm{cris}}^+(C)$ of $\mathcal{O}_C$. Since $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbf{B}_{\mathrm{cris}}^+(C)} = \mathbf{B}_{\mathrm{cris}}^+(C)$

(because the universal vector extension of $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbf{B}^+_{\mathrm{cris}}(C)$ is obtained from the sequence $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ by pushout via $\mathbb{Z}_p \to \mathbf{B}^+_{\mathrm{cris}}(C)$) we obtain a morphism

$$\begin{array}{ccccc} T_pX_x \otimes_{\mathbb{Z}_p} \mathbf{B}^+_{\mathrm{cris}}(C) & \longrightarrow & \mathbb{D}(X_x)_{\mathbf{B}^+_{\mathrm{cris}}(C)} & \xleftarrow{\ \sim\ } & D \otimes_{K_0} \mathbf{B}^+_{\mathrm{cris}}(C)\,. \\ \lambda \otimes a & \longmapsto & \mathbb{D}(\lambda)(a) & & \end{array}$$

Here the isomorphism on the right arises from the quasi-isogeny ρ_x by the same reasoning as in [28, Proposition 5.15]. By Faltings' [13, Theorem 7] the morphism on the left is injective. Since the elements of T_pX_x are φ-invariant inside $T_pX_x \otimes_{\mathbb{Z}_p} \mathbf{B}^+_{\mathrm{cris}}(C)$ and $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)$ equals $\bigcap_{n\in\mathbb{N}_0} \varphi^n \mathbf{B}^+_{\mathrm{cris}}(C)$ we get a monomorphism

$$T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C) \hookrightarrow D \otimes_{K_0} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)\,.$$

It gives rise to a monomorphism

$$\begin{array}{ccccc} T_pX_x & \hookrightarrow & T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^{]0,1]}(C) & \hookrightarrow & D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]}(C) \\ \lambda & \longmapsto & \lambda \otimes 1 & \longmapsto & \mathbb{D}(\lambda)(1) \end{array}$$

since $\widetilde{\mathbf{B}}^{]0,1]}(C)$ is a flat $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)$-algebra. Consider the morphism $\theta : D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]} \to D \otimes_{K_0} C$. Diagram (6.1) shows that $\theta(T_pX_x) \subset (Fil^0_\mu D_K) \otimes_K C$ and so $T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ is in fact contained in $\mathbf{M}_\mu := \mathbf{M}(D, \varphi_D, Fil^0_\mu D_K)$ by Proposition 4.1. But then $\mathbf{M}_\mu \cong \mathbf{M}(0,1)^{\oplus \dim V}$. Namely, assume $\mathbf{M}_\mu \cong \bigoplus_j \mathbf{M}(c_j, d_j)$ with $c_1 > 0$ (note that $\sum_j c_j = 0$). Since the elements of T_pX_x are φ-invariant and $\mathrm{H}^0_\varphi\big(\mathbf{M}(c_1, d_1)\big) = (0)$ by Proposition 3.5 the projection

$$T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C) \longrightarrow \mathbf{M}(c_1, d_1)$$

is zero. Thus $T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C) \hookrightarrow \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$ but this is impossible since we have $\mathrm{rk}_{\mathbb{Z}_p} T_pX_x = \dim V > \mathrm{rk} \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$. This proves the proposition. □

The problem to determine the image of the period morphism π was first mentioned by Grothendieck [19]. We conjecture that this image is $\breve{\mathcal{F}}^a_b$, see below.

7 Conjectures

Inspired by the analogous theory in equal characteristic (see Hartl [21]) we expect the following to be true.

Conjecture 7.1. *The set $\breve{\mathcal{F}}^a_b$ is the unique largest open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{wa}_b$ on which the tensor functor from $\mathrm{Rep}_{\mathbb{Q}_p} G$ to $\mathbb{Q}_p$-$\underline{\mathrm{Loc}}_{\breve{\mathcal{F}}^a_b}$ with property (1.4) exists.*

Remark 7.2. From the definition of local systems the following is immediate: If U_i is a collection of open $\breve{E}$-analytic subspaces of $\breve{\mathcal{F}}^{wa}_b$, $\mathcal{V}_i$ are local systems of $\mathbb{Q}_p$-vector spaces on U_i, and $f_{ij} : \mathcal{V}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{V}_j|_{U_i \cap U_j}$ are isomorphisms which satisfy the cocycle condition on triple intersections, then there exists a unique local system $\mathcal{V}$ on $\bigcup_i U_i$ with $\mathcal{V}|_{U_i} \cong \mathcal{V}_i$. Thus the existence of a unique largest open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{wa}_b$ as in Conjecture 7.1 is evident. This subset may be empty, though.

Remark 7.3. The conjecture that this open subspace is contained in $\breve{\mathcal{F}}_b^a$ is supported by the following results of Andreatta and Brinon [1, 2]. Consider an affinoid subset U of $\breve{\mathcal{F}}_b^{wa}$ which possesses an affine formal model $\mathrm{Spf}\, R$ which is étale over $\mathrm{Spf}\, W(\mathbb{F}_p^{\mathrm{alg}})\langle T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}\rangle$. Assume there exists a tensor functor from $\mathrm{Rep}_{\mathbb{Q}_p} G$ to $\mathbb{Q}_p$-$\underline{\mathrm{Loc}}_U$ with property (1.4). By Proposition 1.4 this corresponds to a representation $\pi_1(U, \bar{x}) \to G$. Choose a faithful representation $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$ and consider the induced representation $\rho_V : \pi_1(U, \bar{x}) \to \mathrm{GL}(V)$. Assume that ρ_V stabilizes a $\mathbb{Z}_p$-lattice in V. Then by [1, Theorem 7.11] ρ_V corresponds to an étale (φ, Γ)-module $\mathbf{M}$ over $\mathbf{A}_R$ which is overconvergent by [2]. If $\mu \in U$ is an analytic point and C is the completion of an algebraic closure of $\mathcal{H}(\mu)$ the fiber $\mathbf{M}_\mu \otimes \widetilde{\mathbf{B}}(C)$ at x is therefore of the form $\mathbf{M}_\mu^\dagger \otimes_{\widetilde{\mathbf{B}}^\dagger(C)} \widetilde{\mathbf{B}}(C)$. By [24, Proposition 5.5.1] the φ-module $\mathbf{M}_\mu^\dagger \otimes_{\widetilde{\mathbf{B}}^\dagger(C)} \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$ over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$ is isomorphic to $\mathbf{M}(0,1)^{\oplus \dim V}$. Due to (1.4) we have $\mathbf{M}_\mu^\dagger \otimes_{\widetilde{\mathbf{B}}^\dagger(C)} \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C) \cong \mathbf{M}(V_{K_0}, b\cdot\varphi, Fil_\mu^\bullet V_{\mathcal{H}(\mu)})$. Thus U must be contained in $\breve{\mathcal{F}}_b^a$.

Remark 7.4. In the analogous situation in equal characteristic on can in fact construct the φ-module $\mathbf{M}(V_{K_0}, b\cdot\varphi, Fil_\mu^\bullet V_{\mathcal{H}(\mu)})$ not just fiber-wise but as a φ-module on all of $\breve{\mathcal{F}}^{\mathrm{an}}$. Globally over $\breve{\mathcal{F}}_b^a$ this φ-module descends from "$\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$" to an étale φ-module over "$\widetilde{\mathbf{B}}^\dagger$", see [21, Corollary 3.2.3]. The later gives rise to a local system on $\breve{\mathcal{F}}_b^a$, see [21, Theorem 3.4.2]. The reason why this can be proved in equal characteristic whereas it is as yet out of reach in mixed characteristic lies in the limitations of the theory of the field of norms in mixed characteristic. This theory is supposed to relate finite étale coverings of $R[\frac{1}{p}]$ to finite étale coverings of $\mathbf{E}_R$. So far it works in mixed characteristic only for rings R as in Remark 7.3, see [1, 17, 18, 31]. In contrast the equal characteristic analogues of $R[\frac{1}{p}]$ and $\mathbf{E}_R$ coincide and so the theory of the field of norms is trivial in equal characteristic.

Conjecture 7.5. *The set $\breve{\mathcal{F}}_b^a$ is the image of the period morphism viewed as a morphism of $\breve{E}$-analytic spaces. The rational Tate module of the universal p-divisible group on $\mathcal{M}^{\mathrm{an}}$ is a local system of $\mathbb{Q}_p$-vector spaces on $\breve{\mathcal{F}}_b^a$ and $\mathcal{M}$ is identified with the étale covering space of $\breve{\mathcal{F}}_b^a$ of $\mathbb{Z}_p$-lattices inside this local system. The local system gives rise to the desired tensor functor from Conjecture 1.2.*

Remark 7.6. This was shown to be true by de Jong [22, Proposition 7.2] in the Lubin-Tate situation where $\breve{\mathcal{F}}_b^a = \breve{\mathcal{F}}_b^{wa} = \breve{\mathcal{F}}^{\mathrm{an}}$.

Remark 7.7. In equal characteristic the étale φ-module over "$\widetilde{\mathbf{B}}^\dagger$" mentioned in Remark 7.4 even allows to construct the analogue of a p-divisible group on the étale covering space of lattices inside the local system on $\breve{\mathcal{F}}_b^a$, see [21, Theorem 3.4.3]. This étale covering space is isomorphic to the analogue of $\mathcal{M}^{\mathrm{an}}$.

Remark 7.8. We expect that in the situation of Remark 7.3 the representation ρ_V is "crystalline" and induces a Dieudonné crystal over $\mathrm{Spec}\, R/pR$. By [23, Main Theorem 1] one could then associate a p-divisible group over $\mathrm{Spec}\, R/pR$ with this Dieudonné crystal. The filtration $Fil^\bullet$ over $\mathcal{M}\big(R[\frac{1}{p}]\big)$ determines a unique lift to a p-divisible group over $\mathcal{M}\big(R[\frac{1}{p}]\big)$. Thus the inclusion $\mathcal{M}\big(R[\frac{1}{p}]\big) \subset \breve{\mathcal{F}}_b^a$ factors through a map $\mathcal{M}\big(R[\frac{1}{p}]\big) \to \mathcal{M}^{\mathrm{an}}$.

A Berkovich's Analytic Spaces

Let $\mathcal{O}_L$ be a complete valuation ring of rank one which is an extension of $\mathbb{Z}_p$ and let L be its fraction field. We briefly recall Berkovich's [5, 6] theory of *L-analytic spaces*. Let B

be an affinoid L-algebra in the sense of [8, Chapter 6] with L-Banach norm $|\,.\,|$. Berkovich calls these algebras *strictly L-affinoid*.

Definition A.1. An *analytic point* x of B is a semi-norm $|\,.\,|_x : B \to \mathbb{R}_{\geq 0}$ which satisfies:

(a) $|f+g|_x \leq \max\{\,|f|_x, |g|_x\,\}$ for all $f, g \in B$,

(b) $|fg|_x = |f|_x\,|g|_x$ for all $f, g \in B$,

(c) $|\lambda|_x = |\lambda|$ for all $\lambda \in L$,

(d) $|\,.\,|_x : B \to \mathbb{R}_{\geq 0}$ is continuous with respect to the norm $|\,.\,|$ on B.

The set of all analytic points of B is denoted $\mathcal{M}(B)$. On $\mathcal{M}(B)$ one considers the coarsest topology such that for every $f \in B$ the map $\mathcal{M}(B) \to \mathbb{R}_{\geq 0}$ given by $x \mapsto |f|_x$ is continuous. Equipped with this topology, $\mathcal{M}(B)$ is a compact Hausdorff space; see [5, Theorem 1.2.1]. Such a space is called a *strictly L-affinoid space*.

Every morphism $\alpha : B \to B'$ of affinoid L-algebras is automatically continuous and hence induces a continuous morphism $\mathcal{M}(\alpha) : \mathcal{M}(B') \to \mathcal{M}(B)$ by mapping the semi-norm $B' \to \mathbb{R}_{\geq 0}$ to the composition $B \to B' \to \mathbb{R}_{\geq 0}$. By definition the $\mathcal{M}(\alpha)$ are the *morphisms* in the category of strictly L-affinoid spaces. In particular, for an affinoid subdomain $\operatorname{Sp} B' \subset \operatorname{Sp} B$ this morphism identifies $\mathcal{M}(B')$ with a closed subset of $\mathcal{M}(B)$.

For every analytic point $x \in \mathcal{M}(B)$ we let $\ker|\,.\,|_x := \{\, b \in B : |b|_x = 0\,\}$. It is a prime ideal in B. We define the *(complete) residue field of x* as the completion with respect to $|\,.\,|_x$ of the fraction field of $B/\ker|\,.\,|_x$. It will be denoted $\mathcal{H}(x)$. There is a natural continuous homomorphism $B \to \mathcal{H}(x)$ of L-algebras. Conversely let K be a *complete extension* of L, by which we mean a field extension of L equipped with an absolute value $|\,.\,| : K \to \mathbb{R}_{\geq 0}$ which restricts on L to the norm of L such that K is complete with respect to $|\,.\,|$. Any continuous L-algebra homomorphism $B \to K$ defines on B a semi-norm which is an analytic point.

In [6, §1.2] Berkovich defines the category of *strictly L-analytic spaces*. These spaces are topological spaces which admit an atlas with strictly L-affinoid charts.

The spaces $\mathcal{M}(B)$ for affinoid L-algebras B are examples for strictly L-analytic spaces. Other examples arise from schemes Y which are locally of finite type over L, see [5, §3.4]. Namely if $Y = \mathbb{A}^n_L$ is affine n-space over L the associated strictly L-analytic space $Y^{\mathrm{an}} = (\mathbb{A}^n_L)^{\mathrm{an}}$ consists of all semi-norms on the polynomial ring $L[y_1, \ldots, y_n]$ as in Definition A.1 (a) - (c). The topology on $(\mathbb{A}^n_L)^{\mathrm{an}}$ is the coarsest topology such that for all $f \in L[y_1, \ldots, y_n]$ the map $(\mathbb{A}^n_L)^{\mathrm{an}} \to \mathbb{R}_{\geq 0}, x \mapsto |f|_x$ is continuous. The space $(\mathbb{A}^n_L)^{\mathrm{an}}$ is the union of the increasing sequence of compact polydiscs $\mathcal{M}\big(L\langle p^m y_1, \ldots, p^m y_n\rangle\big)$ of radii $(p^m, \ldots, p^m)$ for $m \in \mathbb{N}_0$. If $Y \subset \mathbb{A}^n_L$ is a closed subscheme of affine n-space with coherent ideal sheaf $\mathcal{J}$, the ideal sheaf $\mathcal{J}\mathcal{O}_{(\mathbb{A}^n_L)^{\mathrm{an}}}$ defines a closed strictly L-analytic subspace Y^{an} of $(\mathbb{A}^n_L)^{\mathrm{an}}$. Finally, if Y is arbitrary and $\{Y_i\}_i$ is a covering of Y by affine open subschemes then one can glue the associated strictly L-affinoid spaces Y_i^{an} to the L-analytic space Y^{an}. Moreover Y^{an} is Hausdorff if and only if the scheme Y is separated, see [5, Theorems 3.4.1 and 3.4.8].

The relation between strictly L-analytic spaces, rigid analytic spaces, and formal schemes is as follows. To every strictly L-analytic space X which is Hausdorff one can associate a quasi-separated rigid analytic space

$$X^{\mathrm{rig}} \;:=\; \{\, x \in X : \;\mathcal{H}(x) \text{ is a finite extension of } L\,\},$$

see [6, §1.6]. Recall that a rigid analytic space is called *quasi-separated* if the intersection of any two affinoid subdomains is a finite union of affinoid subdomains. To describe the subcategories on which the functor $X \mapsto X^{\mathrm{rig}}$ is an equivalence we need the following terminology. A topological Hausdorff space is called *paracompact* if every open covering $\{U_i\}_i$ has a locally finite refinement $\{V_j\}_j$, where *locally finite* means that every point has a neighborhood which meets only finitely many of the V_j. On the other hand an admissible covering of a rigid analytic space is said to be *of finite type* if every member of the covering meets only finitely many of the other members. A rigid analytic space over L is called *quasi-paracompact* if it possesses an admissible affinoid covering of finite type. Similarly we define the notions *of finite type* and *quasi-paracompact* also for (an open covering of) an admissible formal $\mathcal{O}_L$-scheme in the sense of Raynaud [29]; see also [9].

Theorem A.2. *The following three categories are equivalent:*

(a) the category of paracompact strictly L-analytic spaces,

(b) the category of quasi-separated quasi-paracompact rigid analytic spaces over L, and

(c) the category of quasi-paracompact admissible formal $\mathcal{O}_L$-schemes, localized by admissible formal blowing-ups.

Proof. It is shown in [6, Theorem 1.6.1] that $X \mapsto X^{\mathrm{rig}}$ is an equivalence between (a) and (b). The equivalence of (c) with (b) is due to Raynaud. See [7, Theorem 2.8/3] for a proof. □

Regarding paracompactness the following result was proved in [21, LemmaA.2.5].

Lemma A.3. *Let L be discretely valued with countable residue field and let X be a strictly L-analytic space. Assume that X is Hausdorff and admits a countable covering by strictly L-analytic spaces. Then every open subset of X is a paracompact strictly L-analytic space.*

This applies in particular if $X = Y^{\mathrm{an}}$ for a separated scheme Y of finite type over L.

References

[1] F. Andreatta: *Generalized ring of norms and generalized (φ, Γ)-modules*, University of Padova, Preprint 2005.

[2] F. Andreatta, O. Brinon: forthcoming.

[3] L. Berger: Représentations p-adiques et équations différentielles, *Invent. Math.* **148** (2002), 219–284.

[4] L. Berger: *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Preprint on `arXiv:math.NT/0406601`, version from June 29th, 2004.

[5] V.G. Berkovich: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs **33**, Amer. Math. Soc., Providence 1990.

[6] V.G. Berkovich: Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. I.H.E.S.* **78** (1993), 5–161.

[7] S. Bosch: *Lectures on Formal and Rigid Geometry*, Preprint **378**, University of Münster, SFB 478–Preprint Series, Münster 2005, available at `http://wwwmath1.uni-muenster.de/sfb/about/publ/heft378.ps`.

[8] S. Bosch, U. Güntzer, R. Remmert: *Non-archimedean analysis*, Grundlehren **261**, Springer-Verlag, Berlin-Heidelberg 1984.

[9] S. Bosch, W. Lütkebohmert: Formal and Rigid Geometry
I. Rigid Spaces, *Math. Ann.* **295** (1993), 291–317.
II. Flattening Techniques, *Math. Ann.* **296** (1993), 403–429.

[10] O. Brinon: *Représentations Galoisiennes p-adiques dans le cas relative*, Thèse de l'Université de Paris 11 (Orsay), 2004.

[11] P. Colmez: *Espaces Vectoriels de dimension finie et répresentations de de Rham*, Preprint 2003, available at `http://www.math.jussieu.fr/~colmez`.

[12] P. Colmez, J.-M. Fontaine: Construction des représentations p-adiques semi-stables, *Invent. Math.* **140** (2000), no. 1, 1–43.

[13] G. Faltings: Integral crystalline cohomology over very ramified valuation rings, *J. Amer. Math. Soc.* **12** (1999), no. 1, 117–144.

[14] J.-M. Fontaine: Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in *Journées de géometrie algebrique de Rennes (III), Astérisque* **65** (1979), 3–80.

[15] J.-M. Fontaine: Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, *Ann. of Math. (2)* **115** (1982), no. 3, 529–577.

[16] J.-M. Fontaine: Le corps des périodes p-adiques, *Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque* **223** (1994), 59–111.

[17] J.-M. Fontaine, J.-P. Wintenberger: Le "corps des normes" de certaines extensions algébriques de corps locaux, *C. R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 6, 367–370.

[18] J.-M. Fontaine, J.-P. Wintenberger: Extensions algébrique et corps des normes des extensions APF des corps locaux, *C. R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 8, 441–444.

[19] A. Grothendieck: Groupes de Barsotti-Tate et cristaux, *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pp. 431–436, Gauthier-Villars, Paris, 1971.

[20] A. Grothendieck: *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures 45, Presses de l'Université de Montréal, Montréal 1974.

[21] U. Hartl: *Period Spaces for Hodge Structures in Equal Characteristic*, Preprint on `arXiv:math.NT/0511686`.

[22] J. de Jong: Étale Fundamental groups of non-Archimedean analytic spaces, *Comp. Math.* **97** (1995), 89–118.

[23] J. de Jong: Crystalline Dieudonné module theory via formal and rigid geometry, *Inst. Hautes Études Sci. Publ. Math.* **82** (1995), 5–96.

[24] K. Kedlaya: Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.

[25] R. Kottwitz: Isocrystals with additional structure, *Compositio Math.* **56** (1985), no. 2, 201–220.

[26] Y.I. Manin: The theory of commutative formal groups over fields of finite characteristic, *Russian Math. Surveys* **18** No. 6 (1963), 1–83.

[27] W. Messing: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, LNM **264**, Springer-Verlag, Berlin-New York, 1972.

[28] M. Rapoport, T. Zink: *Period Spaces for p-divisible Groups*, Ann. Math. Stud. **141**, Princeton University Press, Princeton 1996.

[29] M. Raynaud: Géométrie analytique rigide d'apres Tate, Kiehl, ..., *Bul. Soc. Math. France, Mémoires* **39/40** (1974), 319 - 327.

[30] J.-P. Serre: Groupes algébriques associés aux modules de Hodge-Tate, *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)*, Vol. III, pp. 155–188, Astérisque **65**, Soc. Math. France, Paris 1979.

[31] J.-P. Wintenberger: Le corps des normes de certaines extensions infinies de corps locaux; applications, *Ann. Sci. École Norm. Sup. (4)* **16** (1983), no. 1, 59–89.

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